(7) A mass of 1kg is attached to a spring, which causes the spring to stretch 5cm. The spring has a damping constant of 3kg/sec. The mass is subjected to an external force of $10\cos(t)$ Newtons. Write the energy of the system as a function of time.
(8) A mass of 25g is attached to an undamped spring which causes the spring to stretch 1cm. The mass is subjected to an external force of 5 dynes (grams per second squared). Write the energy of the system as a function of time.
(9) A mass of 25g is attached to an undamped spring which causes the spring to stretch 1cm. The mass is subjected to an external force of $5\cos(20t)$ dynes (grams per second squared). Write the energy of the system as a function of time.
(10) A mass of 1kg is attached to an undamped spring which causes the spring to stretch 5cm. The mass is subject to an external force of $2\cos(14t)$ Newtons. Write the energy of the system as a function of time.

§14 Systems of Equations

The final topic we will cover this semester is systems of equations. For example, suppose $x(t)$ and $y(t)$ are both functions of t and satisfy the system of equations

$$\begin{aligned} \frac{dx}{dt} &= 3x + 5y \qquad & x(0) = 1 \\ \frac{dy}{dt} &= 3x + y \qquad & y(0) = 2. \end{aligned}$$

Systems of first order equations arise in population dynamics when you deal with two interacting populations, traditionally foxes and rabbits. Systems of second order equations arise in electrical systems when you have a circuit with more than one loop and in spring-mass systems when you have several different springs connected together. There are a variety of techniques for solving systems of linear differential equations such as the example above. One way is to use some algebraic manipulations to convert the first order linear system to a second order linear equation for x which you can solve by the techniques of chapter 2.

EXAMPLE: Find the general solution to

$$\begin{aligned} \frac{dx}{dt} &= 3x + 5y \\ \frac{dy}{dt} &= 3x + y \end{aligned}$$

Step 1: Solve the first equation for y in terms of x and dx/dt.

$$y = \frac{1}{5}\left(\frac{dx}{dt} - 3x\right)$$

Step 2: Substitute this in for y in both places in the second equation.

$$\frac{1}{5}\frac{d}{dt}\left(\frac{dx}{dt} - 3x\right) = 3x + \frac{1}{5}\left(\frac{dx}{dt} - 3x\right)$$

$$\frac{d^2x}{dt^2} - 3\frac{dx}{dt} = 15x + \frac{dx}{dt} - 3x$$

$$\frac{d^2x}{dt^2} - 4\frac{dx}{dt} - 12x = 0$$

Step 3: Solve this second order linear equation in x.

$$(D^2 - 4D - 12)x = 0$$
$$(D-6)(D+2)x = 0$$
$$x(t) = c_1e^{6t} + c_2e^{-2t}$$

Step 4: Then solve for y

$$\begin{aligned} y(t) &= \frac{1}{5}\left(\frac{d}{dt}(c_1e^{6t} + c_2e^{-2t}) - 3(c_1e^{6t} + c_2e^{-2t})\right) \\ &= \frac{1}{5}\left(6c_1e^{6t} - 2c_2e^{-2t} - 3c_1e^{6t} - 3c_2e^{-2t}\right) \\ &= \frac{3}{5}c_1e^{6t} - c_2e^{-2t} \end{aligned}$$

Note that in some cases it may be better to solve for x in terms of y and the derivatives of y first.

EXAMPLE: Solve the initial value problem

$$\frac{dx}{dt} = 3x - 4y \qquad x(0) = 1$$
$$\frac{dy}{dt} = 2x - y \qquad y(0) = -2$$

FIRST: Find the general solution.

Step 1: Solve the second equation for x in terms of y and y'.

$$x = \frac{1}{2}y' + \frac{1}{2}y$$

Step 2: Substitute this formula into the first equation.

$$\begin{aligned}
\frac{d}{dt}\left(\frac{1}{2}y' + \frac{1}{2}y\right) &= 3\left(\frac{1}{2}y' + \frac{1}{2}y\right) - 4y \\
\frac{1}{2}y'' + \frac{1}{2}y' &= \frac{3}{2}y' - \frac{5}{2}y \\
y'' + y' &= 3y' - 5y \\
y'' - 2y' + 5y &= 0
\end{aligned}$$

Step 3: Solve the second order equation for $y(t)$.

Writing the equation in operator form we get $(D^2 - 2D + 5)y = 0$. The roots of $D^2 - 2D + 5$ are $1 \pm 2i$ so (in the best form for initial value problems)

$$y(t) = C_1 e^t \cos(2t) + C_2 e^t \sin(2t)$$

Step 4: We now solve for $x(t)$.

$$\begin{aligned}
x(t) &= \frac{1}{2}y' + \frac{1}{2}y \\
&= \frac{1}{2}(C_1 e^t \cos(2t) - 2C_1 e^t \sin(2t) + C_2 e^t \sin(2t) + 2C_2 e^t \cos(2t)) \\
&\qquad + \frac{1}{2}(C_1 e^t \cos(2t) + C_2 e^t \sin(2t)) \\
&= (C_1 + C_2)e^t \cos(2t) + (C_2 - C_1)e^t \sin(2t)
\end{aligned}$$

SECOND: Plug in initial values and solve for the constants.

$$\begin{aligned}
x(0) = C_1 + C_2 &\overset{\text{set}}{=} 1 \\
y(0) = C_1 &\overset{\text{set}}{=} -2
\end{aligned}$$

from which we easily get $C_1 = -2$ and $C_2 = 3$. So the solution to the initial value problem is

$$\begin{aligned}
x(t) &= e^t \cos(2t) + 5e^t \sin(2t) \\
y(t) &= -2e^t \cos(2t) + 3e^t \sin(2t)
\end{aligned}$$

Similar manipulations are possible with higher order equations.

EXAMPLE:

$$\frac{d^2x}{dt^2} = -x + 2y$$
$$\frac{dy}{dt} = 5x - 4y$$

Step 1: We solve the second equation for x in terms of y to get

$$x = \frac{1}{5}y' + \frac{4}{5}y$$

Step 2: Now substitute this in for x in both places of the first equation and simplify.

$$\frac{d^2}{dt^2}\left(\frac{1}{5}y' + \frac{4}{5}y\right) = -\left(\frac{1}{5}y' + \frac{4}{5}y\right) + 2y$$
$$\frac{1}{5}y''' + \frac{4}{5}y'' = -\frac{1}{5}y' + \frac{6}{5}y$$
$$y''' + 4y'' = -y' + 6y$$
$$y''' + 4y'' + y' - 6y = 0$$

Step 3: Solve this third order linear equation in y.

$$(D^3 + 4D^2 + D - 6)y = 0$$
$$(D-1)(D+2)(D+3)y = 0$$
$$y(t) = C_1e^t + C_2e^{-2t} + C_3e^{-3t}$$

Step 4: Then solve for x.

$$\begin{aligned} x(t) &= \frac{1}{5}y' + \frac{4}{5}y \\ &= \frac{1}{5}\left(C_1e^t - 2C_te^{-2t} - 3C_3e^{-3t}\right) + \frac{4}{5}\left(C_1e^t + C_2e^{-2t} + C_3e^{-3t}\right) \\ &= C_1e^t + \frac{2}{5}C_2e^{-2t} + \frac{1}{5}C_3e^{-3t} \end{aligned}$$

Exercises:

Find the general solutions

(1) $$\frac{dx}{dt} = y, \quad \frac{dy}{dt} = x - 2y$$

(2) $$\frac{dx}{dt} = 5x + y, \quad \frac{dy}{dt} = -x + y$$

(3) $$\frac{dx}{dt} = 2x - 4y, \quad \frac{dy}{dt} = 3x - 5y$$

(4) $$\frac{dx}{dt} = 3x + 2y, \quad \frac{dy}{dt} = x + 2y$$

Solve the initial value problems

(5)
$$\frac{dx}{dt} = 4x + 3y \qquad x(0) = 1$$
$$\frac{dy}{dt} = -2x - y \qquad y(0) = 1$$

(6)
$$\frac{dx}{dt} = 2x + 1y \qquad x(0) = 1$$
$$\frac{dy}{dt} = -4x - 3y \qquad y(0) = 1$$

(7)
$$\frac{dx}{dt} = x \qquad x(0) = 1$$
$$\frac{dy}{dt} = 2x \qquad y(0) = 2$$

(8)
$$\frac{dx}{dt} = -3x + y \qquad x(0) = -1$$
$$\frac{dy}{dt} = y \qquad y(0) = 2$$

(9)
$$\frac{d^2x}{dt^2} = -2x + y \qquad x(0) = 1$$
$$\frac{dy}{dt} = -6x + 3y \qquad y(0) = 1$$

(10)
$$\frac{d^2x}{dt^2} = 5x + 4y \qquad x(0) = 1$$
$$\frac{dy}{dt} = -2x - y \qquad y(0) = 1$$

§15 Numerical Methods

We studied numerical methods, Euler's method and the improved Euler's method, for approximating the solutions of first order differential equations. The same techniques, with the obvious modifications, work for first order systems as well. First we discuss Euler's method. Given that $y(a) = b$ and $y'(a) = f$ then we can find the tangent line to y(x) and compute the tangent line approximation $y(a+h) \approx b + h \times f$. In an initial value problem we are given initial values of the form $y(a) = b$ and we can use the differential equation to compute $y'(a)$ and then use the tangent line approximation to approximate $y(a+h)$ where h is the step size. Then we repeat the tangent line approximation using our approximation for $y(a+h)$ to approximate $y(a+2h)$ and so on. For systems we use the same techniques, we just have to work out tangent line approximations for each function in our system.

EXAMPLE: Approximate the solution of the first order system

$$\frac{dx}{dt} = 3x + 5y \qquad x(0) = 1$$
$$\frac{dy}{dt} = 3x + y \qquad y(0) = 1$$

at $t = 0.2$ using Euler's method with a step size of 0.1.

$$
\begin{aligned}
x(.1) &\approx x(0) + x'(0) \times 0.1 \\
&= x(0) + (3x(0) + 5y(0)) \times 0.1 \\
&= 1 + 8 \times 0.1 = 1.8 \\
y(.1) &\approx y(0) + y'(0) \times 0.1 \\
&= y(0) + (3x(0) + y(0)) \times 0.1 \\
&= 1 + 4 \times 0.1 = 1.4 \\
x(.2) &\approx x(.1) + x'(.1) \times 0.1 \\
&= x(.1) + (3x(.1) + 5y(.1)) \times 0.1 \\
&\approx 1.8 + 12.4 \times 0.1 = 3.04 \\
y(.2) &\approx y(.1) + y'(.1) \times 0.1 \\
&= y(.1) + (3x(.1) + y(.1)) \times 0.1 \\
&\approx 1.4 + 6.8 \times 0.1 = 2.08
\end{aligned}
$$

Next we discuss the improved Euler's method. This was previously mentioned in extra credit 2, but don't panic if you didn't do extra credit 2. Euler's method approximates the secant line from $(a, y(a))$ to $(a+h, y(a+h))$ by the tangent line at the left endpoint. This works but is quite inefficient. While the slope of the tangent line at the left endpoint is a reasonable guess for the slope of the secant line, a better estimate for the slope of the secant line is to average the slopes of the tangent lines at both the right and left endpoints. Unfortunately, we don't know the right endpoint, indeed that is what we are trying to find. But we do have an approximation to the right endpoint, generated by Euler's method. We can take the slope of the tangent line at this approximate right endpoint, and average it with the slope of the tangent line at the left endpoint to get a new approximation for the slope of the line from $(a, y(a))$ to $(a + h, y(a + h))$. So we first make an estimate for $y(a + h)$, then we improve the estimate. We extend this to systems just as we extended Euler's method to systems, by approximating all the different functions in our system.

EXAMPLE: Approximate the solution of the first order system

$$
\begin{aligned}
\frac{dx}{dt} &= 3x + 5y \qquad & x(0) = 1 \\
\frac{dy}{dt} &= 3x + y \qquad & y(0) = 1
\end{aligned}
$$

at $t = 0.2$ using the improved Euler's method with a step size of 0.1.

$$\begin{aligned}
x(.1) &\approx x(0) + x'(0) \times 0.1 \\
&= x(0) + (3x(0) + 5y(0)) \times 0.1 \\
&= 1 + 8 \times 0.1 = 1.8 \\
y(.1) &\approx y(0) + y'(0) \times 0.1 \\
&= y(0) + (3x(0) + y(0)) \times 0.1 \\
&= 1 + 4 \times 0.1 = 1.4
\end{aligned}$$

Now we improve our estimate at $t = 0.1$

$$\begin{aligned}
x(.1) &\approx x(0) + \frac{x'(0) + x'(.1)}{2} \times 0.1 \\
&= x(0) + \frac{3x(0) + 5y(0) + 3x(.1) + 5y(.1)}{2} \times 0.1 \\
&\approx 1 + \frac{3 + 5 + 5.4 + 7}{2} \times 0.1 = 1.97 \\
y(.1) &\approx y(0) + \frac{y'(0) + y'(.1)}{2} \times 0.1 \\
&= y(0) + \frac{3x(0) + y(0) + 3x(.1) + y(.1)}{2} \times 0.1 \\
&\approx 1 + \frac{3 + 1 + 5.4 + 1.4}{2} \times 0.1 = 1.54
\end{aligned}$$

Now we estimate $x(.2)$ and $y(.2)$

$$\begin{aligned}
x(.2) &\approx x(.1) + x'(.1) \times 0.1 \\
&= x(.1) + (3x(.1) + 5y(.1)) \times 0.1 \\
&\approx 1.97 + 13.61 \times 0.1 = 3.331 \\
y(.2) &\approx y(.1) + y'(.1) \times 0.1 \\
&= y(.1) + (3x(.1) + y(.1)) \times 0.1 \\
&= 1.54 + 7.45 \times 0.1 = 2.285
\end{aligned}$$

Now we improve our estimate at $t = 0.2$

$$\begin{aligned}
x(.2) &\approx x(.1) + \frac{x'(.1) + x'(.2)}{2} \times 0.1 \\
&= x(.1) + \frac{3x(.1) + 5y(.1) + 3x(.2) + 5y(.2)}{2} \times 0.1 \\
&\approx 1.97 + \frac{5.91 + 7.7 + 9.993 + 11.425}{2} \times 0.1 = 3.7214 \\
y(.2) &\approx y(.1) + \frac{y'(.1) + y'(.2)}{2} \times 0.1 \\
&= y(.1) + \frac{3x(.1) + y(.1) + 3x(.2) + y(.2)}{2} \times 0.1 \\
&\approx 1.54 + \frac{5.91 + 1.54 + 9.993 + 2.285}{2} \times 0.1 = 2.5264
\end{aligned}$$

Using the improved Euler's method is more work than using Euler's method, but the answers we get are much closer to the true values $x(.2) = 3.982566142\ldots$ and $y(.2) = 2.657667704\ldots$. That neither answer is particularly close is because our step size of 0.1 is quite large. We would get more accurate answers with a smaller step size (at least until the step size became so small that roundoff error became a problem).

The true usefulness of these procedures for first order systems is that just as we solved a first order system by converting it to a second order equation in the last section, we can convert a second order equation to a first order system and use these techniques to numerically approximate the solution.

EXAMPLE: Convert the second order equation

$$\frac{d^2x}{dt^2} + 4\frac{dx}{dt} + x = 0$$

to a first order system.

Step 1: Introduce a new variable for the first derivative. In this case we let $y(t) = dx/dt$.

Step 2: Rewrite the second derivative of x as the first derivative of the new variable. In this case $d^2x/dt^2 = dy/dt$.

Step 3: Write the original equation and the definition of the new variable as a first order system.

$$\begin{aligned}
\frac{dx}{dt} &= y \\
\frac{dy}{dt} &= -x - 4y
\end{aligned}$$

Now if you wanted to approximate the solution of the original second order equation numerically you could apply Euler's method or the improved Euler's method to the new first order system. This technique will work with equations with variable coefficients and most non-linear equations as well. The general technique for numerically approximating solutions of differential equations is to convert them to first order systems and then use first order techniques (though more often more complicated and accurate techniques like Runge-Kutta methods than Euler's method or the improved Euler's method). For example, matlab requires you to rewrite differential equations as a first order system in order to use the built in commands **ode23** and **ode45** to numerically approximate the solutions. You may get into matlab and type **help ode23** for more information about the format required to use the built in methods (which are Runge-Kutta-Fehlberg methods) if you are interested.

EXERCISES:

Approximate $x(.2)$ and $y(.2)$ for the following systems using Euler's method with a step size of $h = 0.1$. How do the approximate values compare to the true value (which you found in the last section).

(1) $$\frac{dx}{dt} = 4x + 3y \qquad x(0) = 1$$
$$\frac{dy}{dt} = -2x - y \qquad y(0) = 1$$

(2) $$\frac{dx}{dt} = 2x + 1y \qquad x(0) = 1$$
$$\frac{dy}{dt} = -4x - 3y \qquad y(0) = 1$$

(3) $$\frac{dx}{dt} = x \qquad x(0) = 1$$
$$\frac{dy}{dt} = 2x \qquad y(0) = 2$$

(4) $$\frac{dx}{dt} = -3x + y \qquad x(0) = -1$$
$$\frac{dy}{dt} = y \qquad y(0) = 2$$

Approximate $x(.2)$ and $y(.2)$ for the following systems using the improved Euler's method with a step size of $h = 0.1$. How do the approximate values compare to the true values (which you found in the last section)?

(5) $\dfrac{dx}{dt} = 4x + 3y \qquad x(0) = 1$
$\dfrac{dy}{dt} = -2x - y \qquad y(0) = 1$

(6) $\dfrac{dx}{dt} = 2x + 1y \qquad x(0) = 1$
$\dfrac{dy}{dt} = -4x - 3y \qquad y(0) = 1$

(7) $\dfrac{dx}{dt} = x \qquad x(0) = 1$
$\dfrac{dy}{dt} = 2x \qquad y(0) = 2$

(8) $\dfrac{dx}{dt} = -3x + y \qquad x(0) = -1$
$\dfrac{dy}{dt} = y \qquad y(0) = 2$

Rewrite the following second order equations as first order systems.

(9) $x'' + 5x' + 7x = 0$

(10) $x'' + 3x' - x = 0$

(11) $x'' - 3x' + 2x = e^t$

(12) $x'' + x' + 5x = 2\cos(t)$

(13) $x'' + 4tx' - x^2 = t + 1$

(14) $x'' + (x')^2 - tx = 0$

§16 Review Problems

Find the general solutions.

(1) $y'' - y = 0$

(2) $y'' + 6y' + 9y = 0$

(3) $y'' + 3y' + 4y = 0$

(4) $y''' + 7y'' + 14y' + 8y = 0$

(5) $y'' + 2y' - 3y = e^{2x}$

(6) $y'' + 2y' - 5y = \cos(5x)$

(7) $y'' - 3y' - 10y = x + e^x$

(8) $y'' - 3y' + y = \sin(2x) - \cos(2x)$

(9) $y'' + 4y' + 4y = e^{-2x}$

(10) $y'' + y' + 10y = \cos^2(x)$

Solve the initial value problems.

(11) $y'' + 4y' + 3y = 0, \qquad y(0) = 1, \quad y'(0) = 0$

(12) $y'' + 4y' - 5y = e^{2x}, \qquad y(0) = 1, \quad y'(0) = 1$

(13) $y'' - 8y' + 12y = x^2, \qquad y(0) = 3, \quad y'(0) = -2$

(14) $y'' + 6y' + 8y = \sin(x), \qquad y(0) = -1, \quad y'(0) = 0$

(15) $y'' - 5y' - 14y = e^{7x}, \qquad y(0) = -2, \quad y'(0) = -1$

(16) $y'' + 3y' - 4y = e^{2x}\cos(3x), \qquad y(0) = 1, \quad y'(0) = 0$

(17) A mass of 1kg is attached to an undamped spring, stretching the spring 40cm. The mass is pulled down 20cm and released. What is the equation of the resulting motion? Graph the motion.
(18) A mass of 2kg is attached to a spring, pulling it down 25cm. The damping constant of the spring is 0.5kg/sec. The mass is then pushed up 10cm and released. What is the equation of the resulting motion? Graph the motion.
(19) A mass of 120g is attached to an undamped spring, pulling it down 5cm. What is the resonant frequency of this spring?
(20) An electric circuit has a coil with inductance 2 henrys, a resistor of 120 ohms, and a 2 microfarad capacitor. The circuit has an impressed voltage of $170\cos(120\pi t)$. What is the steady state current of the circuit?

Find the general solutions

(21) $$\frac{dx}{dt} = 2x \qquad \frac{dy}{dt} = x - 2y$$

(22) $$\frac{dx}{dt} = 3y \qquad \frac{dy}{dt} = 2x + y$$

(23) $$\frac{dx}{dt} = -4x + 3y \qquad \frac{dy}{dt} = -2x + y$$

(24) $$\frac{dx}{dt} = -15x + y \qquad \frac{dy}{dt} = 59x - 6y$$

Solve the initial value problems

(25) $$\frac{dx}{dt} = 4x + 2y \quad x(0) = 1 \qquad \frac{dy}{dt} = -x + y \quad y(0) = 1$$

(26) $$\frac{dx}{dt} = 7x + 17y \quad x(0) = 1 \qquad \frac{dy}{dt} = 3x + y \quad y(0) = 1$$

Approximate $x(.2)$ and $y(.2)$ for the following systems using Euler's method with a step size of $h = 0.1$. How do the approximate values compare to the true values found above?

(27) $$\frac{dx}{dt} = 4x + 2y \quad x(0) = 1 \qquad \frac{dy}{dt} = -x + y \quad y(0) = 1$$

(28) $$\frac{dx}{dt} = 7x + 17y \quad x(0) = 1 \qquad \frac{dy}{dt} = 3x + y \quad y(0) = 1$$

Approximate $x(.2)$ and $y(.2)$ for the following systems using the improved Euler's method with a step size of $h = 0.1$. How do the approximate values compare to the true values found above?

(29) $$\frac{dx}{dt} = 4x + 2y \qquad x(0) = 1$$
$$\frac{dy}{dt} = -x + y \qquad y(0) = 1$$

(30) $$\frac{dx}{dt} = 7x + 17y \qquad x(0) = 1$$
$$\frac{dy}{dt} = 3x + y \qquad y(0) = 1$$