(31) $y'' + 2y' - 3y = 0, \quad y(0) = 1, \quad y'(0) = 0$

(32) $y'' - 3y' + 2y = 0, \quad y(0) = 0, \quad y'(0) = 1$

(33) $y'' - 2y' - 8y = 0, \quad y(2) = 1, \quad y'(2) = 0$

(34) $y''' + 2y'' - 5y' - 6y = 0, \quad y(0) = 1, \quad y'(0) = 0, \quad y''(0) = 0$

(35) $y'' - 4y' - 21y = 0, \quad y(1) = 0, \quad y'(1) = 1$

(36) $y'' + y' - \frac{3}{4}y = 0, \quad y(1) = 1, \quad y'(1) = 0$

(37) $y'' + 2y' + y = 0, \quad y(0) = 1, \quad y'(0) = 2$

(38) $y''' + 6y'' + 9y' = 0, \quad y(0) = 1, \quad y'(0) = 0, \quad y''(0) = 0$

(39) $y''' + 4y'' + 3y' = 0, \quad y(0) = 3, \quad y'(0) = -3, \quad y''(0) = 11$

(40) $y'''' - 8y'' + 16y = 0, \quad y(0) = 0, \quad y'(0) = 0, \quad y''(0) = 0, \quad y'''(0) = 0$

(41) $y''' + y'' + y' + y = 0$

(42) $y''' - y'' + y' - y = 0$

(43) $y''' + 4y'' + 6y' + 4y = 0$

(44) $y'''' + 8y'' + 16y = 0$

(45) $y''' + 5y'' + 4y' - 10y = 0, \quad y(0) = 1, \quad y'(0) = 0, \quad y''(0) = 0$

§7 Inhomogeneous Equations

Discussion: Now that we have learned to solve homogeneous equations, the next topic is inhomogeneous equations. From the discussion in section 5, we know that all we need to find the general solution to an inhomogeneous equation is any one particular solution to the equation along with the general solution to the homogeneous equation. The main method we have to find such a particular solution is called *undetermined coefficients*. The reason it is called undetermined coefficients is that *guessing* is not nearly so impressive a name. The basic idea is that if we apply a constant coefficient linear differential operator to the function e^{rx} for any constant r, the result is just Be^{rx} for some constant B. So if we want to solve $Ly = Be^{rx}$, we should guess the solution is of the form $y = Ae^{rx}$ for some constant A. Having made this guess we plug it into the equation $Ly = Be^{rx}$ and solve for A to find our particular solution. Of course, this guess won't always work. Then we guess again using somewhat more sophisticated guesses. Undetermined coefficients qualifies as a method rather than just a trick because it is possible to write down a precise procedure for coming up with exactly the right guess. But doing so right now would be more confusing than helpful. So we will instead deal with the simplest case first and then explain what to do if the obvious guess fails later in the examples.

Paradigm: $y'' + 4y' + 3y = 4e^{-2x}$

STEP 1: Find the general solution to the homogeneous equation, $y_h(x)$.

We obtain $D^2 + 4D + 3 = 0$ which has roots -1 and -3. So the general solution to the homogeneous equation is $y_h(x) = c_1e^{-x} + c_2e^{-3x}$.

STEP 2: Guess the form of the particular solution $y_p(x)$.

As indicated in the discussion above, a good guess for the form of the particular solution is $y_p(x) = Ae^{-2x}$.

STEP 3: Plug the guess for the particular solution, $y_p(x)$, into the equation and solve for the undetermined coefficients.

We have

$$\begin{aligned} y_p(x) &= Ae^{-2x} \\ y_p'(x) &= -2Ae^{-2x} \\ y_p''(x) &= 4Ae^{-2x} \end{aligned}$$

So plugging into the equation we find

$$\begin{aligned} y_p''(x) + 4y_p'(x) + 3y_p(x) &= 4Ae^{-2x} + 4(-2Ae^{-2x}) + 3(Ae^{-2x}) \\ &= (4A - 8A + 3A)e^{-2x} \\ &= -Ae^{-2x} \overset{\text{set}}{=} 4e^{-2x} \end{aligned}$$

Solving the last equation we get $A = -4$ so $y_p(x) = -4e^{-2x}$.

STEP 4: The general solution is now $y_p(x) + y_h(x)$.

The general solution to the equation is

$$y(x) = -4e^{-2x} + c_1e^{-x} + c_2e^{-3x}$$

That is a very simple process. We now consider some of the situations where things can go wrong.

EXAMPLE: $y'' + 4y' + 3y = 2e^{-x}$

Step 1: $D^2 + 4D + 3 = 0$ still has roots -1 and -3 and the general solution to the homogeneous equation is still $y_h(x) = c_1 e^{-x} + c_2 e^{-3x}$.

Step 2: Here the right hand side has the form Be^{rx} where $r = -1$ is one of the roots of the homogeneous equation. Suppose we try our guess $y_p(x) = Ae^{-x}$ anyway. Then $y_p'(x) = -Ae^{-x}$ and $y_p''(x) = Ae^{-x}$ and so plugging into the equation yields

$$\begin{aligned} y_p'' + 4y_p' + 3y_p &= Ae^{-x} + 4(-Ae^{-x}) + 3(Ae^{-x}) \\ &= (A - 4A + 3A)e^{-x} \\ &= 0 \overset{\text{set}}{=} 2e^{-x} \end{aligned}$$

which is a problem. The trouble here is that since e^{-x} is part of the homogeneous solution, it drops out. What we need is a second term of the form e^{-x} which won't drop out. Looking back at section 6, we see that if we have a double root we have a term of the form Axe^{rx} so let's try that in our present problem.

Step 2: (Take 2) Guess $y_p(x) = Axe^{-x}$

Step 3:

$$\begin{aligned} y_p(x) &= Axe^{-x} \\ y_p'(x) &= Ae^{-x} - Axe^{-x} \\ y_p''(x) &= -2Ae^{-x} + Axe^{-x} \end{aligned}$$

So plugging into the equation we obtain

$$\begin{aligned} y_p'' + 4y_p' + 3y_p &= (-2Ae^{-x} + Axe^{-x}) + 4(Ae^{-x} - Axe^{-x}) + 3(Axe^{-x}) \\ &= (-2A + 4A)e^{-x} + (A - 4A + 3A)xe^{-x} \\ &= 2Ae^{-x} \overset{\text{set}}{=} 2e^{-x} \end{aligned}$$

Solving the last equation we find $A = 1$ and $y_p(x) = xe^{-x}$.

Step 4: $y(x) = xe^{-x} + c_1 e^{-x} + c_2 e^{-3x}$.

There are two points to be noted about the last example. One is that if your first guess doesn't work, try try again. There is no penalty for making a wrong guess, beyond wasting a little bit of time, so try your guess and if it doesn't work, guess again. The second point is that you can often get some insight into the need for a second guess from what goes wrong with the first guess. In particular from this example, if you end up with an equation of the form $0 = Be^{rx}$, you probably need to multiply your initial guess by x.

EXAMPLE: $y'' + 4y' + 4y = e^{-2x}$

Step 1: We obtain $D^2 + 4D + 4 = (D+2)^2 = 0$ so we have a double root of -2 and the general solution of the homogeneous equation is $y_h(x) = c_1e^{-2x} + c_2xe^{-2x}$.

Step 2: We don't want to guess Ae^{-2x} because that is part of the homogeneous solution. But so is Axe^{-2x}. So let's try $y_p(x) = Ax^2e^{-2x}$.

Step 3:

$$\begin{aligned} y_p(x) &= Ax^2e^{-2x} \\ y_p'(x) &= -2Ax^2e^{-2x} + 2Axe^{-2x} \\ y_p''(x) &= 4Ax^2e^{-2x} - 8Axe^{-2x} + 2Ae^{-2x} \end{aligned}$$

So plugging into the equation yields

$$\begin{aligned} y_p'' + 4y_p' + 4y_p &= (4Ax^2e^{-2x} - 8Axe^{-2x} + 2Ae^{-2x}) \\ &\qquad + 4(-2Ax^2e^{-2x} + 2Axe^{-2x}) + 4(Ax^2e^{-2x}) \\ &= (4A - 8A + 4A)x^2e^{-2x} + (-8A + 8A)xe^{-2x} + 2Ae^{-2x} \\ &= 2Ae^{-2x} \stackrel{\text{set}}{=} e^{-2x} \end{aligned}$$

Solving the last equation we find $A = 1/2$ and so $y_p = 1/2x^2e^{-2x}$.

Step 4: $y(x) = 1/2x^2e^{-2x} + c_1e^{-2x} + c_2xe^{-2x}$.

Now we consider some right hand sides which are not quite of the form Be^{rx}, but not far from it.

EXAMPLE: $y'' + 5y' + 6y = 2xe^x$.

Step 1: $D^2 + 5D + 6 = 0$ has roots -2 and -3 so $y_h(x) = c_1e^{-2x} + c_2e^{-3x}$.

Step 2: Well, the first guess that probably comes to your mind is $y_p(x) = Axe^x$, so let's give that a try.

Step 3:

$$\begin{aligned} y_p(x) &= Axe^x \\ y_p'(x) &= Axe^x + Ae^x \\ y_p''(x) &= Axe^x + 2Ae^x \end{aligned}$$

Plugging into our equation we obtain

$$\begin{aligned} y_p'' + 5y_p' + 6y_p &= (Axe^x + 2Ae^x) + 5(Axe^x + Ae^x) + 6(Axe^x) \\ &= (A + 5A + 6)xe^x + (2A + 5A)e^x \\ &= 12Axe^x + 7Ae^x \overset{\text{set}}{=} 2xe^x \end{aligned}$$

Well that almost worked out. Unfortunately, we got that extra $7Ae^x$ term. So how should we get rid of it? But no real harm done, we just have to guess again. And we use the fact that we got an extra $7Ae^x$ term to guess that we need a Be^x term to go along with our Axe^x term. That makes sense. Whenever we've had an xe^x term in the past in any solution we had an e^x to go along with it.

Step 2: (Take 2) $y_p = Axe^x + Be^x$.

Step 3: (Take 2)

$$\begin{aligned} y_p(x) &= Axe^x + Be^x \\ y_p'(x) &= Axe^x + (A + B)e^x \\ y_p''(x) &= Axe^x + (2A + B)e^x \end{aligned}$$

Plugging into our equation we obtain

$$\begin{aligned} y_p'' + 5y_p' + 6y_p &= (Axe^x + (2A + B)e^x) + 5(Axe^x + (A + B)e^x) + 6(Axe^x + Be^x) \\ &= (A + 5A + 6A)xe^x + (2A + B + 5A + 5B + 6B)e^x \\ &= 12Axe^x + (7A + 12B)e^x \overset{\text{set}}{=} 2xe^x \end{aligned}$$

From the last equation we find $12A = 2$ and $7A + 12B = 0$, so $A = 1/6$ and $B = -7/72$. Hence $y_p(x) = 1/6xe^x - 7/72e^x$.

Step 4: $y(x) = 1/6xe^x - 7/72e^x + c_1e^{-2x} + c_2e^{-3x}$.

Before we consider the next example, we state and prove an easy and useful theorem (the best kind).

THEOREM. *Suppose L is a linear operator and $Ly = f$ and $Lz = g$. Then $L(y+z) = f+g$.*

PROOF: This follows immediately from the definition of linearity. $L(y + z) = Ly + Lz = f + g$.

EXAMPLE: $y'' - 2y' - 8y = e^x - e^{-x}$

Step 1: $D^2 - 2D - 8 = 0$ has roots -2 and 4 so $y_h(x) = c_1 e^{-2x} + c_2 e^{4x}$.

Step 2: Using the previous theorem, we can now split the hunt for a particular solution into two parts.

Step 2: (Part 1) Find a particular solution for $y'' - 2y' - 8y = e^x$. We guess $y_{p1}(x) = Ae^x$.

Step 3: (Part 1)

$$\begin{aligned} y_{p1}(x) &= Ae^x \\ y'_{p1}(x) &= Ae^x \\ y''_{p1}(x) &= Ae^x \end{aligned}$$

Plugging into our equation we obtain

$$\begin{aligned} y''_{p1} - 2y'_{p1} - 8y_{p1} &= Ae^x - 2(Ae^x) - 8(Ae^x) \\ &= (A - 2A - 8A)e^x \\ &= -9Ae^x \overset{\text{set}}{=} e^x \end{aligned}$$

From the last equation we find $A = -1/9$ so $y_{p1}(x) = -1/9e^x$.

Step 2: (Part 2) Find a particular solution for $y'' - 2y' - 8y = -e^{-x}$. We guess $y_{p2}(x) = Ae^{-x}$.

Step 3: (Part 2)

$$\begin{aligned} y_{p2}(x) &= Ae^{-x} \\ y'_{p2}(x) &= -Ae^{-x} \\ y''_{p2}(x) &= Ae^{-x} \end{aligned}$$

Plugging into our equation we obtain

$$\begin{aligned} y''_{p2} - 2y'_{p2} - 8y_{p2} &= Ae^{-x} - 2(-Ae^{-x}) - 8(Ae^{-x}) \\ &= (A + 2A - 8A)e^{-x} \\ &= -5Ae^{-x} \overset{\text{set}}{=} -e^{-x} \end{aligned}$$

From the last equation we find $A = 1/5$ so $y_{p2}(x) = 1/5e^{-x}$.

Step 3: (Conclusion) $y_p(x) = y_{p1}(x) + y_{p2}(x) = -1/9e^x + 1/5e^{-x}$.

Step 4: $y(x) = -1/9e^x + 1/5e^{-x} + c_1 e^{-2x} + c_2 e^{4x}$.

Finally we consider how to handle sines and cosines on the right hand side. There are two different approaches and I will give examples of both. Most students find the real variable method easier than the complex method, but be warned that in later courses you may discover the complex method easier in many applications.

We will begin with the real variables approach. The key idea is to figure out what to guess if the right hand side is a sine and/or cosine. The key result is that whenever your right hand side is of the form $a\sin(\omega x)+b\cos(\omega x)$, guess a particular solution in the form $y_p(x) = A\sin(\omega x)+B\cos(\omega x)$. It needs to be emphasized that you must guess $y_p(x)$ with a sine/cosine *pair*, even if the original right hand side just has an individual sine or cosine. Just because a or b is 0 doesn't mean either A or B will be.

EXAMPLE: $y'' + 3y' + 2y = \sin(2x)$

Step 1: We obtain $D^2 + 3D + 2 = 0$ which has roots -1 and -2. So the general solution to the homogeneous equation is $y_h(x) = c_1e^{-x} + c_2e^{-2x}$.

Step 2: Since the right hand side is of the form $\sin(2x)$, we guess $y_p(x) = A\sin(2x) + B\cos(2x)$.

Step 3:

$$\begin{aligned} y_p(x) &= A\sin(2x) + B\cos(2x) \\ y_p'(x) &= 2A\cos(2x) - 2B\sin(2x) \\ y_p''(x) &= -4A\sin(2x) - 4B\cos(2x) \end{aligned}$$

Plugging into our equation we obtain

$$\begin{aligned} y_p'' + 3y_p' + 2y_p &= (-4A - 6B + 2A)\sin(2x) + (-4B + 6A + 2B)\cos(2x) \\ &= (-2A - 6B)\sin(2x) + (6A - 2B)\cos(2x) \stackrel{\text{set}}{=} \sin(2x) \end{aligned}$$

This gives us two equations in two unknowns,

$$\begin{aligned} -2A - 6B &= 1 \\ 6A - 2B &= 0 \end{aligned}$$

Solving the pair of equations we get $A = -1/20$ and $B = -3/20$, so the particular solution is $y_p(x) = -1/20\sin(2x) - 3/20\cos(2x)$.

Step 4: $y(x) = -1/20\sin(2x) - 3/20\cos(2x) + c_1e^{-x} + c_2e^{-2x}$.

We will go over the complex method next. We start with a theorem similar to that in the previous section.

THEOREM. *If a, b and c are real and $ay''+by'+cy=f$, then $a(\Re[y])''+b(\Re[y])'+c(\Re[y])=\Re[f]$ and $a(\Im[y])''+b(\Im[y])'+c(\Im[y])=\Im[f]$.*

PROOF: Take the real and imaginary parts of each side of the given equation $ay''+by'+cy=f$.

This gives us a strategy to use with sines and cosines. We write them as the real part of a complex exponential and then find a particular solution for the complex equation. Finally we take the real part of the complex particular solution to find the particular solution to the original real equation. This differs enough from the first paradigm that I will write it down as a new paradigm.

Paradigm: $y''+y'-2y=-\cos(2x)$.

STEP 1: Find the general solution of the homogeneous equation.

The roots of $D^2+D-2=0$ are -2 and 1 so the general solution of the homogeneous equation is $y_h(x)=c_1e^{-2x}+c_2e^x$.

STEP 2: Write the right hand side of the equation as the real part of a complex exponential.

$-\cos(2x)=\Re[-e^{i2x}]$.

STEP 3: Find a particular solution to the corresponding complex equation.

Let $\tilde{y}_p$ denote a particular solution to $\tilde{y}_p''+2\tilde{y}_p'-\tilde{y}_p=-e^{i2x}$. Then we use the techniques developed earlier in this section to find $\tilde{y}_p(x)$.

SubStep 1: Guess the form of $\tilde{y}_p$

$\tilde{y}_p=Ae^{2ix}$.

SubStep 2: Plug into the equation and solve for the undetermined coefficients.

$$\begin{aligned}\tilde{y}_p&=Ae^{2ix}\\ \tilde{y}_p'&=2iAe^{2ix}\\ \tilde{y}_p''&=-4Ae^{2ix}\end{aligned}$$

Plugging into the equation we obtain

$$\begin{aligned}\tilde{y}_p'' + \tilde{y}_p' - 2\tilde{y}_p &= (-4Ae^{2ix}) + (2iAe^{2ix}) - 2(Ae^{2ix}) \\ &= (-4A + 2iA - 2A)e^{2ix} \\ &= (-6 + 2i)Ae^{2ix} \overset{\text{set}}{=} -e^{2ix}\end{aligned}$$

Solving the last equation we find $A = -1/(-6+2i) = 3/20 + 1/20i$. So $\tilde{y}_p(x) = (3/20 + 1/20i)e^{2ix}$.

STEP 4: Take the real part of the complex particular solution to find the real particular solution.

$y_p(x) = \Re[\tilde{y}_p(x)] = 3/20\cos(2x) - 1/20\sin(2x)$.

STEP 5: The general solution is $y(x) = y_p(x) + y_h(x)$.

$y(x) = 3/20\cos(2x) - 1/20\sin(2x) + c_1e^{-2x} + c_2e^x$.

We illustrate a few more possible difficulties in the final two examples.

EXAMPLE: $y'' + 3y' + 2y = 3e^{-2x}\sin(x)$.

Step 1: The roots of $D^2 + 3D + 2 = 0$ are -1 and -2 so the homogeneous solution is $y_h(x) = c_1e^{-x} + c_2e^{-2x}$.

Step 2: $3e^{-2x}\sin(x) = \Re[-3ie^{(-2+i)x}]$. (Note that we have to put the $-3i$ to get a 3 after multiplying by $i\sin(x)$ in the identity for complex exponentials. It would also be possible to write the real function as the imaginary part of a complex exponential rather than the real part. In that case, we would have to take the imaginary part of the complex particular solution in step 4.)

Step 3:

SubStep 1: $\tilde{y}_p(x) = Ae^{(-2+i)x}$

SubStep 2:

$$\begin{aligned}\tilde{y}_p(x) &= Ae^{(-2+i)x} \\ \tilde{y}_p'(x) &= (-2+i)Ae^{(-2+i)x} \\ \tilde{y}_p''(x) &= (3-4i)Ae^{(-2+i)x}\end{aligned}$$

Plugging into our equation we obtain

$$\begin{aligned}\tilde{y}_p'' + 3\tilde{y}_p' + 2\tilde{y}_p &= [(3-4i)Ae^{(-2+i)x}] + 3[(-2+i)Ae^{(-2+i)x}] + 2[Ae^{(-2+i)x}] \\ &= (3-4i-6+3i+2)Ae^{(-2+i)x} \\ &= (-1-i)Ae^{(-2+i)x} \overset{\text{set}}{=} -3ie^{(-2+i)x}\end{aligned}$$

Solving the last equation we find $A = -3i/(-1-i) = 3/2 + 3/2i$. So $\tilde{y}_p(x) = (3/2 + 3/2i)e^{(-2+i)x}$.

Step 4: $y_p(x) = e^{-2x}(3/2\cos(x) - 3/2\sin(x))$.

Step 5: $y(x) = e^{-2x}(3/2\cos(x) - 3/2\sin(x)) + c_1 e^{-x} + c_2 e^{-2x}$.

EXAMPLE: $y'' + y = \cos(x) + 2\sin(x)$.

Step 1: The roots of $D^2 + 1 = 0$ are $\pm i$ so the homogeneous solution is $y_h(x) = c_1\cos(x) + c_2\sin(x)$.

Step 2: We could split this into two pieces, but since both terms have the same frequency we can do it all in one by writing $\cos(x) + 2\sin(x) = \Re[(1-2i)e^{ix}]$.

Step 3:

SubStep 1: $\tilde{y}_p(x) = Axe^{ix}$. Since e^{ix} already appears in the homogeneous solution, we have to multiply by x.

SubStep 2:

$$\begin{aligned}\tilde{y}_p(x) &= Axe^{ix} \\ \tilde{y}_p'(x) &= iAxe^{ix} + Ae^{ix} \\ \tilde{y}_p''(x) &= -Axe^{ix} + 2iAe^{ix}\end{aligned}$$

Plugging into our equation we obtain

$$\begin{aligned}\tilde{y}_p'' + \tilde{y}_p &= (-Axe^{ix} + 2iAe^{ix}) + (Axe^{ix}) \\ &= (-A + A)xe^{ix} + 2iAe^{ix} \\ &= 2iAe^{ix} \overset{\text{set}}{=} (1-2i)e^{ix}\end{aligned}$$

Solving the last equation we find $A = (1-2i)/(2i) = -1 - 1/(2i) = -1 + (1/2)i$. So $\tilde{y}_p(x) = (-1 + (1/2)i)xe^{ix}$.

Step 4: $y_p(x) = -x\cos(x) + (1/2)x\sin(x)$.

Step 5: $y(x) = -x\cos(x) + 1/2x\sin(x) + c_1\cos(x) + c_2\sin(x)$.

After all these examples, the reader may have discovered the following general rules.

• If the right-hand side is of the form e^{rx} and e^{rx} is not already in the homogeneous solution, then the guess should be Ae^{rx}.
• If the right-hand side is of the form $(a_nx^n + \cdots a_1x + a_0)e^{rx}$ with $a_n \neq 0$ and e^{rx} is not already in the homogeneous solution, then the guess should be $(A_nx^n + \cdots A_1x + A_0)e^{rx}$. Note that while $a_n \neq 0$, the other a_j terms may or may not be 0 in this case.
• If the right-hand side is of the form $(a_nx^n + \cdots a_1x + a_0)e^{rx}$ with $a_n \neq 0$ and x^me^{rx} is in the homogeneous solution, then the guess should be $x^{m+1}(A_nx^n + \cdots A_1x + A_0)e^{rx}$.
• If the right-hand side is of the form $a\sin(\omega x) + b\cos(\omega x)$, guess $A\sin(\omega x) + B\cos(\omega x)$. Note that you have to include both A and B even if a or b is 0.
• You can also replace sines and cosines with complex exponentials to apply the above rules to them.

Of course, if you misapply these rules all that will happen is that you will get stuck as we did in the examples above where a wrong initial guess was made. In that case you just reconsider and guess again.

Exercises:

(1) $y'' + 3y' + 2y = e^x$

(2) $y'' - 5y' + 6y = -e^{4x}$

(3) $y'' - 4y' + 3y = e^{2x} - e^{-2x}$

(4) $y'' + y' - 12y = xe^x$

(5) $y'' + 2y' - 3y = e^{-3x}$

(6) $y'' + 2y' + y = xe^{-x} - 2e^{3x}$

(7) $y'' + 4y' + 3y = \cos(2x)$

(8) $y'' + 7y' + 12y = e^x\cos(3x)$

(9) $y'' - 2y' - 15y = \sin(x) - \cos(x)$

(10) $y'' + y = \cos(x)$

(11) $y'' + 5y' + 4y = e^{-x}, \qquad y(0) = 0, \quad y'(0) = 0$

(12) $y'' - 3y' - 4y = 3e^{-2x}, \qquad y(0) = 1, \quad y'(0) = 0$

(13) $y'' + 4y' + 4y = e^{-x} + 2e^{2x}, \qquad y(0) = 0, \quad y'(0) = 1$

(14) $y'' - y = \sin(x), \qquad y(0) = 0, \quad y'(0) = 0$

(15) $y'' + 2y' + 2y = e^{x}(1 + \cos(x)), \qquad y(0) = 3, \quad y'(0) = 0$

(16) $y'' + 3y' + 2y = e^{3x}, \qquad y(0) = 0, \quad y'(0) = 0$

(17) $y'' + y = \cos(x), \qquad y(0) = 1, \quad y'(0) = 1$

(18) $y'' + 2y' + 2y = e^{x}, \qquad y(0) = 1, \quad y'(0) = 0$

(19) $y'' + 2y' + y = e^{-x}, \qquad y(0) = 0, y'(0) = 0$

(20) $y''' - y' = e^{2x}, \qquad y(0) = 1, \quad y'(0) = 0, \quad y''(0) = 0$

For problems 21 through 25, guess the form of the particular solution, but don't solve the equations.

(21) $y'' + y' + y = e^{x}\sin(2x) - x^2$

(22) $y'' - y' + 3y = e^{2x} - xe^{x} + \cos(x)$

(23) $y'' + y = e^{x}\sin(2x) - e^{x}\cos(3x - \pi/7)$

(24) $y'' - y = x^3 e^{x} + x^2\cos(x)$

(25) $y'' + 4y' + y = x^2 e^{-3x}\cos(2x) + xe^{-3x}\sin(2x)$

§8 Variation of Parameters

Discussion: The method of undetermined coefficients is the most efficient way to solve an inhomogeneous equation if we can guess the proper form for the particular solution. Unfortunately we can't always guess the proper form. If we can find the general solution to the homogeneous equation, the method of variation of parameters will always work. Because the method is complicated, we will only cover the second order case. Consider the following example

$$y'' + y = \tan(x)$$

We first find two linearly independent solutions to the homogeneous equation $y'' + y = 0$. In this case two linearly independent solutions are $\cos(x)$ and $\sin(x)$. We now guess that the solution to the inhomogeneous equation will take the form

$$y(x) = u(x)\cos(x) + v(x)\sin(x)$$

Differentiating this yields

$$y'(x) = -u(x)\sin(x) + v(x)\cos(x) + u'(x)\cos(x) + v'(x)\sin(x)$$

We should now differentiate this again to compute y'' but this is getting ugly. y'' will have 6 terms. At this point I observe I have 2 degrees of freedom in my guess for the solution (i.e. 2 unknown functions) and I only need to satisfy one equation. With 2 unknown functions I should be able to satisfy 2 equations and I will now tack on the additional condition that besides solving the original problem, $y'' + y = \tan(x)$, I will choose u and v so that

$$u'(x)\cos(x) + v'(x)\sin(x) = 0.$$

Using this I can then get

$$y'(x) = -u(x)\sin(x) + v(x)\cos(x)$$

and I then differentiate again to get

$$y''(x) = -u(x)\cos(x) - v(x)\sin(x) - u'(x)\sin(x) + v'(x)\cos(x)$$

So plugging into $y'' + y = \tan(x)$ we get

$$y'' + y = \begin{array}{c} -\,u(x)\cos(x) - v(x)\sin(x) - u'(x)\sin(x) + v'(x)\cos(x) \\ +\,u(x)\cos(x) + v(x)\sin(x) \end{array} \overset{\text{set}}{=} \tan(x)$$

From which we get the equation

$$-u'(x)\sin(x) + v'(x)\cos(x) = \tan(x)$$

We now have two equations for $u'(x)$ and $v'(x)$.

$$\begin{aligned} u'(x)\cos(x) + v'(x)\sin(x) &= 0 \\ -u'(x)\sin(x) + v'(x)\cos(x) &= \tan(x) \end{aligned} \tag{1}$$

Multiplying the first equation by $\sin(x)$ and the second equation by $\cos(x)$ and adding we obtain

$$\begin{aligned} v'(x)(\sin^2(x) + \cos^2(x)) &= \sin(x) \\ v'(x) &= \sin(x) \\ v(x) &= -\cos(x) + C_1 \end{aligned}$$

Multiplying the first equation in (1) by $\cos(x)$ and the second equation in (1) by $-\sin(x)$ and adding we obtain

$$\begin{aligned} u'(x)(\cos^2(x) + \sin^2(x)) &= -\sin^2(x)/\cos(x) \\ u'(x) &= -\sin^2(x)/\cos(x) \\ u(x) &= \sin(x) - \log\left(\frac{\sin(x)+1}{\cos(x)}\right) + C_2 \end{aligned}$$

Finally we get

$$\begin{aligned} y(x) &= \sin(x)\cos(x) - \log\left(\frac{\sin(x)+1}{\cos(x)}\right)\cos(x) + C_2\cos(x) - \cos(x)\sin(x) + C_1\sin(x) \\ &= -\log\left(\frac{\sin(x)+1}{\cos(x)}\right)\cos(x) + C_2\cos(x) + C_1\sin(x) \end{aligned}$$

In this one case, I *do not* advise you to learn the procedure for solving inhomogeneous equations using variation of parameters. The manipulations given above are complicated and most students have lots of trouble with them. Using them, I can derive the following formula and I advise you to just copy down the formula on your crib sheet and use it whenever necessary.

THEOREM. *If $y_1(x)$ and $y_2(x)$ are two linearly independent solutions to the second order linear homogeneous equation $y''(x) + b(x)y'(x) + c(x)y(x) = 0$ then*

$$y(x) = -y_1(x)\int_{x_0}^{x} \frac{y_2(s)g(s)}{W(y_1,y_2)(s)}\,ds + y_2(x)\int_{x_0}^{x} \frac{y_1(s)g(s)}{W(y_1,y_2)(s)}\,ds + C_1y_1(x) + C_2y_2(x)$$

is the general solution to the inhomogeneous equation $y''(x) + b(x)y'(x) + c(x)y(x) = g(x)$ where x_0, C_1 and C_2 are arbitrary constants and $W(y_1, y_2)(s)$ is the Wronskian.

This looks at first glance to have three arbitrary constants but in fact it only has two. Any given solution can be obtained with every choice of x_0 by making the right choice of C_1 and C_2. The advantage of this form is that while we often won't be able to evaluate the integrals, we will always be able to evaluate any definite integral from x_0 to x_0. So if we have an initial value problem with the values given at x_0, we will be able to work out the values of C_1 and C_2 even if we can't compute the integrals directly.

Paradigm: Solve $x^2y'' + xy' - 4y = x$ given that two linearly independent solutions of the corresponding homogeneous equation are $y_1(x) = x^2$ and $y_2(x) = x^{-2}$.

STEP 1: Find two linearly independent solutions of the corresponding homogeneous problem.

Here we are given that two solutions are $y_1(x) = x^2$ and $y_2(x) = x^{-2}$.

STEP 2: Evaluate the solution of the inhomogeneous problem using the formula.

$$W(y_1, y_2)(s) = s^2 \times -2s^{-3} - 2s \times s^{-2} = -4s^{-1}$$

Pick $x_0 = 0$ for convenience

$$\int_0^x \frac{s^{-2}s}{-4s^{-1}}\, ds = \int_0^x -\frac{1}{4}\, ds = -x/4$$
$$\int_0^x \frac{s^2 s}{-4s^{-1}}\, ds = \int_0^x -\frac{s^4}{4}\, ds = -x^5/20$$

So we get

$$\begin{aligned} y(x) &= -x^2 \times -x/4 + x^{-2} \times -x^5/20 + C_1x^2 + C_2x^{-2} \\ &= x^3/5 + C_1x^2 + C_2x^{-2} \end{aligned}$$

EXAMPLE: Solve the initial value problem

$$y'' + 6y' + 5y = \sqrt{x}, \qquad y(1) = 0, \quad y'(1) = 0$$

FIRST: Find the general solution

Step 1: The homogeneous equation is $y'' + 6y' + 5y = 0$

SubStep 1: $(D^2 + 6D + 5)y = (D + 5)(D + 1)y = 0$

SubStep 2: roots are -5 and -1

SubStep 3: The general solution is $C_1e^{-5x} + C_2e^{-x}$ and e^{-5x} and e^{-x} are two linearly independent solutions of the homogeneous equation.

Step 2:

$$W(e^{-5x}, e^{-x}) = e^{-5x} \times -e^{-x} - (-5e^{-5x} \times e^{-x}) = 4e^{-6x}$$

We choose $x_0 = 1$ since we are given initial values at 1. Then

$$\int_1^x \frac{e^{-s}\sqrt{s}}{4e^{-6s}}\, ds = \int_1^x \frac{e^{5s}\sqrt{s}}{4}\, ds$$
$$\int_1^x \frac{e^{-5s}\sqrt{s}}{4e^{-6s}}\, ds = \int_1^x \frac{e^{s}\sqrt{s}}{4}\, ds$$

I can evaluate neither of these integrals. So the best I can do for the general solution to the inhomogeneous problem is

$$y(x) = -e^{-5x}\int_1^x \frac{e^{5s}\sqrt{s}}{4}\, ds + e^{-x}\int_1^x \frac{e^{s}\sqrt{s}}{4}\, ds + C_1 e^{-5x} + C_2 e^{-x}$$

SECOND: Plug in the initial values and solve for the constants.

$$\begin{aligned} y(1) &= -e^{-5} \times 0 + e^{-1} \times 0 + C_1 e^{-5} + C_2 e^{-1} \overset{\text{set}}{=} 0 \\ y'(x) &= 5e^{-5x}\int_1^x \frac{e^{5s}\sqrt{s}}{4}\, ds - e^{-5x}\frac{e^{5x}\sqrt{x}}{4} \\ &\quad - e^{-x}\int_1^x \frac{e^{s}\sqrt{s}}{4}\, ds + e^{-x}\frac{e^{x}\sqrt{x}}{4} \\ &\quad - 5C_1 e^{-5x} - C_2 e^{-x} \\ y'(1) &= 5e^{-5} \times 0 - \frac{1}{4} - e^{-1} \times 0 + \frac{1}{4} - 5C_1 e^{-5} - C_2 e^{-1} \overset{\text{set}}{=} 0 \end{aligned}$$

Here we have used the fact that the derivative of the integral is the original function. So the initial values yield the two equations

$$\begin{aligned} C_1 e^{-5} + C_2 e^{-1} &= 0 \\ -5C_1 e^{-5} - C_2 e^{-1} &= 0 \end{aligned}$$

These have the solution $C_1 = 0$ and $C_2 = 0$ so our solution to the initial value problem is

$$y(x) = -e^{-5x}\int_1^x \frac{e^{5s}\sqrt{s}}{4}\, ds + e^{-x}\int_1^x \frac{e^{s}\sqrt{s}}{4}\, ds$$

Exercises:

(1) $y'' + y = \sec(x)$ (2) $y'' + 4y = \sin(2x)\cos(2x)$

(3) $y'' - y = \sqrt{x}$ (4) $y'' + 2y' + 2y = \log(x)$

(5) $y'' - 4y' + 3y = \tan(x), \quad y(0) = 0, \quad y'(0) = 0$

§9 Boundary Value Problems

A second order equation will have two arbitrary constants in its general solution, so if you want to specify a particular solution you should expect to require two additional conditions. So far, the conditions we have considered have always been initial values, where you are given the value of the function and its derivative at a point. Physically, this corresponds to knowing the position and velocity at a given time. But it is quite possible to be given other types of conditions. Another common example is boundary conditions, where you are told the value of the function at two different points. Physically, this corresponds to knowing the position at two different times. At first glance, there appears to be little different about such problems. You just find the general solution, then plug in the two conditions and solve to find the values of the constants. This is indeed how you go about solving the problem. But there turn out to be various interesting difficulties that arise in boundary value problems that don't arise in initial value problems.

EXAMPLE: Solve the boundary value problem $y'' + y = 0$, $y(0) = 1$, $y(1) = 2$.

FIRST: Find the general solution. This is a homogeneous equation so it is easy to find that the general solution is $y(x) = c_1 \sin(x) + c_2 \cos(x)$.

SECOND: Plug in the boundary values and solve for the constants.

$$y(0) = c_1 \sin(0) + c_2 \cos(0) = c_2 \overset{\text{set}}{=} 1$$
$$y(1) = c_1 \sin(1) + c_2 \cos(1) \overset{\text{set}}{=} 2$$

Solving the equations we get

$$c_1 = \frac{2 - \cos(1)}{\sin(1)}$$
$$c_2 = 1$$

so the solution is $y(x) = \frac{2-\cos(1)}{\sin(1)} \sin(x) + \cos(x)$.

That didn't seem very exciting. But consider the next example.

EXAMPLE: $y'' + y = 0$, $y(0) = 1$, $y(\pi) = 2$.

FIRST: As before, we find the general solution is $y(x) = c_1 \sin(x) + c_2 \cos(x)$.

SECOND: This time we plug in and get

$$y(0) = c_1 \sin(0) + c_2 \cos(0) = c_2 \overset{\text{set}}{=} 1$$
$$y(\pi) = c_1 \sin(\pi) + c_2 \cos(\pi) = c_2 \overset{\text{set}}{=} 2$$

This can't happen since c_2 can't equal both 1 and 2. In this case there is no solution to the boundary value problem. That never happened with initial value problems, and there is a theorem that it can't happen for any reasonable initial value problem. But boundary value problems are a whole new ball game. Another weird possibility is in the following example.

EXAMPLE: $y'' + y = 0$, $y(0) = 0$, $y(\pi) = 0$.

FIRST: As always, the general solution is $y(x) = c_1 \sin(x) + c_2 \cos(x)$.

SECOND: We plug in and get

$$y(0) = c_1 \sin(0) + c_2 \cos(0) = c_2 \overset{\text{set}}{=} 0$$
$$y(\pi) = c_1 \sin(\pi) + c_2 \cos(\pi) = -c_2 \overset{\text{set}}{=} 0$$

So we see that c_2 emphatically must be 0. But we don't know anything about c_1. Any function of the form $y(x) = c_1 \sin(x)$ satisfies the given boundary value problem. Again, with initial value problems we don't have this difficulty. Any reasonable initial value problem has exactly one solution. With boundary value problems, we might have just one solution, as in the first example, or no solutions, as in the second example, or infinitely many solutions, as in the third example.

Boundary value problems often arise in connection with partial differential equations. In this case, it is usually most interesting to know when you get infinitely many solutions. This isn't quite so standardized a problem that I can give you a paradigm to mimic, but I can work one to give you the general idea.

QUESTION: For what values of k does $y'' + k^2 y = 0$, $y(0) = 0$, $y(\pi) = 0$, have infinitely many solutions?

Observe that $y(x) = 0$ is always a solution to the given problem, but we are interested in when there are other solutions as well (once there is a second solution you are guaranteed that there are infinitely many solutions).

The way to begin such a problem is to write out the general solution and plug in the boundary values. The general solution to $y'' + k^2y = 0$ is $y(x) = c_1 \sin(kx) + c_2 \cos(kx)$. We plug in the boundary values to get

$$y(0) = c_1 \sin(0) + c_2 \cos(0) = c_2 \overset{\text{set}}{=} 0$$
$$y(\pi) = c_1 \sin(k\pi) + c_2 \cos(k\pi) \overset{\text{set}}{=} 0$$

From the first equation we know c_2 must be 0. Plugging this into the second equation we get $c_1 \sin(k\pi) = 0$, so either $c_1 = 0$ or $\sin(k\pi) = 0$. If $c_1 = 0$, we just get the solution $y(x) = 0$, so if we want infinitely many solutions we need $\sin(k\pi) = 0$. This happens exactly when k is an integer.

Exercises:

Solve the following boundary value problems if possible. (1) $y'' + 3y' + 2y = 0$, $y(0) = 0$, $y(1) = 1$.

(2) $y'' + 2y' + 10y = 0$, $y(0) = 1$, $y(1) = 0$.
(3) $y'' + 10y' + 21y = 0$, $y(0) = 4$, $y(1) = -1$.
(4) $y'' + 9y' + 10y = 0$, $y(0) = 0$, $y(\pi) = 5$.
(5) $y'' + y = 0$, $y(0) = 0$, $y(\pi/2) = 0$.
(6) $y'' + 4y = 0$, $y(0) = 0$, $y(\pi/2) = 0$.
(7) $y'' + 9y = 0$, $y(0) = 0$, $y(\pi/2) = 0$.
(8) $y'' + 16y = 0$, $y(0) = 0$, $y(\pi/2) = 0$.
(9) $y'' - y = 0$, $y(0) = 0$, $y(\pi/2) = 0$.
(10) $y'' = 0$, $y(0) = 0$, $y(\pi/2) = 0$.
(11) For what values of k does $y'' + k^2y = 0$, $y(0) = 0$, $y(\pi/2) = 0$ have infinitely many solutions?
(12) For what values of k does $y'' + k^2y = 0$, $y(0) = 0$, $y(1) = 0$ have infinitely many solutions?
(13) For what values of ℓ does $y''+y = 0$, $y(0) = 0$, $y(\ell) = 0$ have infinitely many solutions?
(14) For what values of k does $y'' + k^2y = 0$, $y(0) = 0$, $y(\pi) = 1$ have no solutions?
(15) For what values of ℓ does $y'' + y = 0$, $y(0) = 0$, $y(\ell) = 0$ have no solutions?

§10 Partial Differential Equations

Linear constant coefficient differential equations arise in many places. One of these is from linear constant coefficient partial differential equations. Consider a vibrating string stretched from $x = 0$ to $x = \pi$. The amplitude $u(x, t)$ of the vibration at the point x at the time t satisfies the wave equation,

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$$

where c is a constant which depends on the string. As with all differential equations, one good way is to guess what a solution might look like and then plug in our guess and see what happens. A standard trick for this is "separation of variables." We guess that there are solutions $u(x, t)$ that are products of a function of x alone and a function of t alone, so

$$u(x, t) = X(x)T(t).$$

Now we plug this into our equation. Since we treat t as a constant when we compute the partial derivative with respect to x,

$$\begin{aligned} \frac{\partial u}{\partial x} &= \frac{\partial}{\partial x} X(x)T(t) \\ &= \left(\frac{d}{dx} X(x) \right) T(t) \\ &= X'(x)T(t) \end{aligned}$$

Taking another derivative we see $(\partial^2/\partial x^2)u(x, t) = X''(x)T(t)$. Similarly, since we treat x as a constant when we compute the partial derivative with respect to t we obtain $(\partial^2/\partial t^2)u(x, t) = X(x)T''(t)$. Plugging these into the wave equation and then dividing both sides by $u(x, t) = X(x)T(t)$ we get

$$\begin{aligned} X(x)T''(t) &= c^2 X''(x)T(t) \\ \frac{X(x)T''(t)}{X(x)T(t)} &= c^2 \frac{X''(x)T(t)}{X(x)T(t)} \\ \frac{T''(t)}{T(t)} &= c^2 \frac{X''(x)}{X(x)} \end{aligned}$$

Now the left hand side of the above equation depends only on t and not on x. But this is equal to the right hand side which depends only on x and not on t. When can a function of x and not of t be equal to a function of t and not of x? Only when the function is a constant function. So let both sides of the equation be equal to a constant, λ. Then we have reduced the second order partial differential equation to two second order ordinary differential equations:

$$\frac{T''(t)}{T(t)} = \lambda$$

$$c^2 \frac{X''(x)}{X(x)} = \lambda$$

We can now solve these equations simultaneously. If $\lambda > 0$, then it has a square root say $\sqrt{\lambda} = \mu$ so $\lambda = \mu^2$ and the solutions of the ordinary differential equations are

$$\begin{aligned} T''(t) &= \mu^2 T(t) & c^2 X''(x) &= \mu^2 X(x) \\ T''(t) - \mu^2 T(t) &= 0 & c^2 X''(x) - \mu^2 X(x) &= 0 \\ (D^2 - \mu^2)T(t) &= 0 & (c^2 D^2 - \mu^2)X(x) &= 0 \\ T(t) &= C_1 e^{\mu t} + C_2 e^{-\mu t} & X(x) &= C_3 e^{\mu x/c} + C_4 e^{-\mu x/c} \end{aligned}$$

and so we have a collection of solutions of the partial differential equation of the form

$$u(x,t) = (C_3 e^{\mu x/c} + C_4 e^{-\mu x/c})(C_1 e^{\mu t} + C_2 e^{-\mu t})$$

Of course, it is also possible that λ is negative, in which case we let $\sqrt{-\lambda} = \nu$ so $\lambda = -\nu^2$ and the solutions of the ordinary differential equations are

$$\begin{aligned} T''(t) &= -\nu^2 T(t) & c^2 X''(x) &= -\nu^2 X(x) \\ T''(t) + \nu^2 T(t) &= 0 & c^2 X''(x) + \nu^2 X(x) &= 0 \\ (D^2 + \nu^2)T(t) &= 0 & (c^2 D^2 + \nu^2)X(x) &= 0 \\ T(t) &= C_1 \cos(\nu t) + C_2 \sin(\nu t) & X(x) &= C_3 \cos(\nu x/c) + C_4 \sin(\nu x/c) \end{aligned}$$

and so we have another collection of solutions of the partial differential equations of the form

$$u(x,t) = (C_1 \cos(\nu t) + C_2 \sin(\nu t))(C_3 \cos(\nu x/c) + C_4 \sin(\nu x/c))$$

Finally, if $\lambda = 0$ then the solutions of the ordinary differential equation are

$$\begin{aligned} T''(t) &= 0 & c^2 X''(x) &= 0 \\ T(t) &= C_1 + C_2 t & X(x) &= C_3 + C_4 x \end{aligned}$$

and so we have one last collection of solutions of the partial differential equation of the form

$$u(x,t) = (C_1 + C_2 t)(C_3 + C_4 x)$$

The wave equation is a *linear* partial differential equation, which means that sums of solutions are still solutions, just as for linear ordinary differential equations. So now we can form lots of different solutions out of our various collections of solutions. Usually we will want to find solutions that have particular properties. For example, if our string is tacked down at $x = 0$ and $x = \pi$ so that the amplitude of vibration is always 0 at those points, then we want $u(0, t) = u(\pi, t) = 0$ for all t. This condition will eliminate many of the different solutions given above (see problem 5). It turns out that all solutions of the wave equation satisfying the condition that the endpoints are tacked down can be written as linear combinations of the solutions we have found, so we will have the general solution to the wave equation subject to this condition. But the situation is more complicated than for ordinary differential equations, because while we have eliminated many of the different solutions with our tacking down condition, we still have infinitely many solutions left so our general solution has infinitely many arbitrary constants. But that is a problem best left to a later course in Fourier series, or better yet partial differential equations. For now, we will settle for noting that the linear constant coefficient ordinary differential equations we have been considering in this chapter often come up in solving linear constant coefficient partial differential equations.

Exercises:
(1) Find solutions in the form $u(x, t) = X(x)T(t)$ for the heat equation $\partial u/\partial t = \kappa^2 \partial^2 u/\partial x^2$.
(2) Find solutions in the form $u(x, y) = X(x)Y(y)$ for Laplace's equation $\partial^2 u/\partial x^2 + \partial^2 u/\partial y^2 = 0$.
(3) Find solutions in the form $u(x, t) = X(x)T(t)$ for the equation $\partial u/\partial t = \partial^2 u/\partial x^2 - u$ (heat equation with cooling).
(4) Find solutions in the form $u(x, y) = X(x)Y(y)$ for Poisson's equation $\partial^2 u/\partial x^2 + \partial^2 u/\partial y^2 = u$.
(5) Show that the only solutions of the form $u(x, t) = X(x)T(t)$ to the wave equation that satisfy $u(0, t) = u(\pi, t) = 0$ for all t are $u(x, t) = (C_1 \cos(cnt) + C_2 \sin(cnt)) \sin(nx)$ where n is a positive integer.

§11 Free Motion

Consider a mass on a spring. The forces acting on the mass are gravity with a force mg, where m is the mass and g is the acceleration of gravity (9.8m/sec^2), the restoring force of the spring with a force $-kl$, where k is the spring constant, l is how much the spring is