

Chapter 2

Linear Constant Coefficient Higher Order Equations

§1 COMPLEX VARIABLES

Now that we have learned a variety of techniques for handling first order equations, we will move on to higher order equations. Unfortunately, higher order equations can be a lot trickier. Fortunately, we will avoid most of the difficulties by only concentrating on the easy ones, which are also the most common in basic applications in engineering and physics (and when you get to the non-basic applications you should be ready to learn about the harder ones). The easy equations usually reduce down to solving polynomial equations — college algebra stuff. We won't be concerned with just the real roots of our polynomial equations though, we will have to consider the complex roots as well. And the complex roots we encounter will then have to be used in exponential functions. So before we begin solving the differential equations, we will take a couple of days to go over the theory of complex variables.

We let $i = \sqrt{-1}$. A complex number is a number of the form $a + bi$ where a and b are real numbers. Arithmetic with complex numbers is almost exactly the same as with real numbers except every time you square an i you replace it with -1 . There is also one new operation called conjugation where you replace i by $-i$.

Complex Arithmetic.

Real Part:

$$\Re(a + bi) = a$$

$$\Re(7 - 4i) = 7$$

Imaginary Part:

$$\Im(a + bi) = b$$

$$\Im(3 - 11i) = -11$$

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Conjugation:

$$\overline{(a + bi)} = a - bi$$

$$\overline{(-2 + 5i)} = -2 - 5i$$

Addition:

$$(a + bi) + (c + di) = (a + c) + (b + d)i$$

$$(2 + 3i) + (4 - 7i) = (2 + 4) + (3 - 7)i = 6 - 4i$$

Subtraction:

$$(a + bi) - (c + di) = (a - c) + (b - d)i$$

$$(3 + 2i) - (-1 - 4i) = (3 - -1) + (2 - -4)i = 4 + 6i$$

Multiplication:

Here you just multiply out term by term and collect the like terms.

$$(a + bi)(c + di) = ac + adi + bci + bdi^2$$

$$= ac + adi + bci - bd$$

$$= (ac - bd) + (ad + bc)i$$

$$(3 - 4i)(2 + i) = (6 - -4) + (3 - 8)i = 10 - 5i$$

Division:

This is the one tricky operation. The trick is to multiply the numerator and the denominator by the conjugate of the denominator. This will yield a real denominator and then everything is easy.

$$\begin{aligned} \frac{a + bi}{c + di} &= \frac{(a + bi)\overline{(c + di)}}{(c + di)\overline{(c + di)}} \\ &= \frac{(a + bi)(c - di)}{(c + di)(c - di)} \\ &= \frac{ac + bci - adi + bd}{c^2 + cdi - cdi + d^2} \\ &= \frac{(ac + bd) + (bc - ad)i}{c^2 + d^2} \\ &= \frac{ac + bd}{c^2 + d^2} + \frac{bc - ad}{c^2 + d^2}i \end{aligned}$$

In the last step we use the fact that $c^2 + d^2$ is a real number and we know that the quotient of a sum is the sum of the quotients and that we know how to divide by real numbers. The

fact that when you multiply a complex number by its conjugate you get a real product is very important and will show up again.

$$\begin{aligned}\frac{2-7i}{3+i} &= \frac{(2-7i)(3-i)}{(3+i)(3-i)} \\ &= \frac{(6-7) + (-2-21)i}{(9+1) + (-3+3)i} \\ &= \frac{-1-23i}{10} \\ &= -0.1 - 2.3i\end{aligned}$$

Complex Exponentials.

Now that we can do arithmetic with complex numbers, we want to do calculus with them. We start with the exponential function. The trick is to expand the function in a Taylor series about the point $z = 0$ and see what happens. In the following, t denotes a real variable. You also need to use the following relations: $i^2 = -1$, $i^3 = -i$, $i^4 = 1$, $i^5 = i$, etc.

$$\begin{aligned}e^{it} &= 1 + it + \frac{(it)^2}{2!} + \frac{(it)^3}{3!} + \frac{(it)^4}{4!} + \frac{(it)^5}{5!} + \dots \\ &= 1 + it + \frac{i^2 t^2}{2!} + \frac{i^3 t^3}{3!} + \frac{i^4 t^4}{4!} + \frac{i^5 t^5}{5!} + \dots \\ &= 1 + it - \frac{t^2}{2!} - i \frac{t^3}{3!} + \frac{t^4}{4!} + i \frac{t^5}{5!} - \dots \\ &= \left(1 - \frac{t^2}{2!} + \frac{t^4}{4!} - \dots\right) + i \left(t - \frac{t^3}{3!} + \frac{t^5}{5!} - \dots\right) \\ &= \cos(t) + i \sin(t)\end{aligned}$$

as we recognize the Taylor series about 0 for cosine and sine. Using this relationship, we can now define the exponential of a general complex number as follows:

$$e^{x+iy} = e^x e^{iy} = e^x (\cos(y) + i \sin(y))$$

We will use this relation often in solving differential equations.

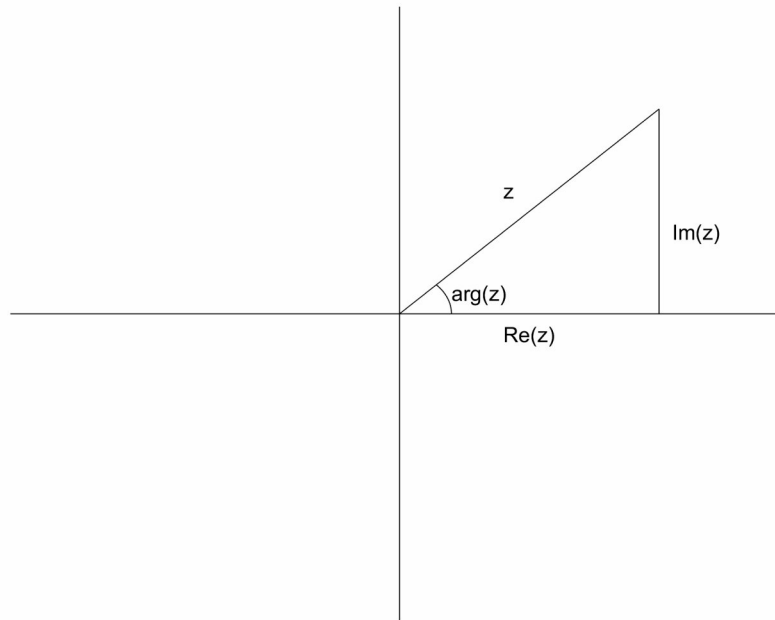
Exercises: Evaluate the following expressions. Write your answers in the form $a + bi$.

- | | |
|------------------------------------|--|
| (1) $(2 + 4i) + (3 - 5i)$ | (2) $(-3 + i) + (4 + 7i)$ |
| (3) $(4 - 3i) - (-2 + 5i)$ | (4) $(3 + 5i) - (5 - 6i)$ |
| (5) $(1 - i)(3 + 2i)$ | (6) $(1 + i)(1 - i)$ |
| (7) $(-3 - 2i)(-4 + 5i)$ | (8) $(4 - 2i)(3 + 2i)$ |
| (9) $\frac{2 - i}{3 + i}$ | (10) $\frac{3 - i}{2 + 3i}$ |
| (11) $\frac{-3 + 2i}{5 - 2i}$ | (12) $\frac{-1 - 3i}{-2 + 3i}$ |
| (13) $e^{i\pi}$ | (14) $e^{-i\frac{\pi}{3}}$ |
| (15) e^{2-i} | (16) $e^{3-i\frac{\pi}{6}}$ |
| (17) $(2 + i)e^{1+i\frac{\pi}{2}}$ | (18) $\frac{e^{1+i\frac{2\pi}{3}}}{-2 + 8i}$ |
| (19) $3e^{2-i}e^{4+2i}$ | (20) $\frac{e^{1+i\frac{\pi}{2}}}{e^{1-i\frac{\pi}{2}}}$ |

§2 GEOMETRY OF COMPLEX NUMBERS

In section 1 we covered the basic arithmetic of complex numbers, which we used to solve differential equations in later sections. In this section we will give a geometric interpretation of complex numbers. This interpretation isn't *necessary* to solve differential equations, but most students find it helpful in understanding how complex numbers work.

Complex numbers don't lie on a number line but rather on the complex plane. Complex numbers can be written $x + iy$, where x and y are real. We naturally graph the number $x + iy$ at the point (x, y) . The x -axis is referred to as the Real axis and the y -axis as the Imaginary axis. We can also identify points in the plane using polar coordinates rather than rectangular coordinates. The polar coordinates of a complex number are called the modulus and the argument of the complex number, where modulus is the distance from the origin (r) and the argument is the angle made with the Real axis by the segment from the origin to the point (θ). The modulus and argument of a complex number therefore have the same definitions as the usual conversion from rectangular to polar coordinates (see Figure 0).



Modulus:

$$|x + iy| = \sqrt{x^2 + y^2}$$

Argument:

$$\arg(x + iy) = \begin{cases} \arctan(y/x), & x \geq 0 \\ \arctan(y/x) + \pi, & x < 0, y \geq 0 \\ \arctan(y/x) - \pi, & x < 0, y < 0 \end{cases}$$

Recall that \arctan is a π -periodic function, usually defined to take values from $-\pi/2$ to $\pi/2$. On the other hand, the argument of a complex number is a polar angle, which can range from $-\pi$ to π . That is why we have the somewhat complicated expression for the argument. If you are trying to compute an argument using a calculator, you can punch in $\arctan(y/x)$ but then you are responsible for deciding if you need to adjust by a factor of $\pm\pi$. Matlab, on the other hand, has such adjustments built-in. If you type in the command **angle(2+3i)** you get the result 0.9828, while if you type in the command **angle(-2-3i)** you get the result -2.1588, even though $\arctan(y/x)$ is the same in both cases. Note also that matlab has the value $i = \sqrt{-1}$ built-in. It also defines $j = \sqrt{-1}$, in accordance with common usage in engineering (where i is instantaneous current). One warning: if you have run a program that included the lines $i = 1$ or $j = 0$, then i and j will have been redefined and will not be $\sqrt{-1}$ later during that same session of matlab. If that happens you can just give the command **i=sqrt(-1)** to reset the value of i .

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Once you have gone from rectangular to polar coordinates, the natural thing is to go from polar to rectangular coordinates. Suppose we have a complex number z with modulus $|z|$ and argument $\arg(z)$. Then the real and imaginary parts of z can be determined using the formulas for conversion from polar to rectangular coordinates.

$$\text{Real Part: } \Re(z) = |z| \cos(\arg(z))$$

$$\text{Imaginary Part: } \Im(z) = |z| \sin(\arg(z))$$

We are now ready for a sneaky trick, that turns out to be very useful. If we write $z = \Re(z) + i\Im(z)$ and use the preceding rules for the real and imaginary parts and do a little bit of algebra and remember what we learned about complex exponentials in the previous section and don't worry about having the word "and" too many times in the same sentence we get

$$\begin{aligned} z &= \Re(z) + i\Im(z) \\ &= |z| \cos(\arg(z)) + i|z| \sin(\arg(z)) \\ &= |z|(\cos(\arg(z)) + i \sin(\arg(z))) \\ &= |z|(e^{0+i \arg(z)}) \\ &= |z|e^{i \arg(z)} \end{aligned}$$

This formula is called the polar form of the complex number z . In the next section we will see how converting complex numbers from rectangular form $x + iy$ into polar form $re^{i\theta}$ and back again can simplify trig identities. For now we will concentrate on how understanding the polar form helps us understand the geometric meaning of the basic operations of complex arithmetic. Addition and subtraction of complex numbers are just vector addition and subtraction like you studied in Calculus III with this geometric interpretation of complex numbers as vectors in the plane. Multiplication and division are more complicated.

To understand the multiplication of complex numbers geometrically, it is easiest to work in polar coordinates. Suppose we want to multiply $re^{i\theta}$ and $se^{i\phi}$. We compute

$$\begin{aligned} re^{i\theta} \times se^{i\phi} &= rse^{i\theta} e^{i\phi} \\ &= rse^{i(\theta+\phi)} \end{aligned}$$

So the modulus of the product is the product of the moduli and the argument of the product is the sum of the arguments. In other words, to multiply two complex numbers you multiply the lengths of the corresponding vectors (in polar coordinates) and add the

angles (in polar coordinates). Division is just the reverse of course. You divide the moduli and subtract the angles.

Let us check how this interpretation works for i^2 . The complex number i is represented by the vector $(0, 1)$ which has modulus 1 and argument $\pi/2$, that is to say it is a vector of length 1 and it is at a right angle with the Real axis. When we multiply this number by itself, we multiply the length by itself and $1 \times 1 = 1$ so the product will also have modulus 1. We add the argument to itself, and twice a right angle is a straight angle, so the vector should point in the negative real direction. And the vector of length 1 in the negative real direction represents the number -1 , so we do indeed have $i^2 = -1$.

§3 ALTERNATE FORMS FOR TRIGONOMETRIC FUNCTIONS

Discussion: Quick, what does the graph of $3 \cos(2x) + 4 \sin(2x)$ look like? This is a difficult question for most people to answer. While you know what the graphs of trig functions look like, adding them usually requires plotting both graphs and then trying to add the heights. It turns out that using some trig identities you can rewrite $3 \cos(2x) + 4 \sin(2x)$ in the form $A \cos(\omega x - \phi)$ for the right choices of A , ω and ϕ . This is just a cosine wave with amplitude A , circular frequency ω , and phase shift ϕ , so it is easier to visualize. But most students have difficulty remembering how the trig identities go, even when they are in trig, let alone several years later. Using complex polar coordinates, it is relatively easy to convert sums of cosine and sine functions to this nicer form.

The key to rewriting $3 \cos(2x) + 4 \sin(2x)$ using complex numbers is the following pair of facts: $\Re(3e^{2xi}) = 3 \cos(2x)$ and $\Re(-4ie^{2xi}) = 4 \sin(2x)$. So

$$3 \cos(2x) + 4 \sin(2x) = \Re(3e^{2xi} - 4ie^{2xi}) = \Re((3 - 4i)e^{2xi})$$

We now rewrite $3 - 4i$ in polar form (using the results from §2).

$$r = |3 - 4i| = \sqrt{3^2 + 4^2} = 5 \quad \theta = \arctan(-4/3) \approx -.927$$

So $3 - 4i \approx 5e^{-.927i}$. Now we plug this into our representation for $3 \cos(2x) + 4 \sin(2x)$ and

obtain

$$\begin{aligned} 3 \cos(2x) + 4 \sin(2x) &= \Re((3 - 4i)e^{2xi}) \\ &\approx \Re(5e^{-.927i}e^{2xi}) \\ &= \Re(5e^{(2x-.927)i}) \\ &= 5 \cos(2x - .927) \end{aligned}$$

We are now ready to write a paradigm for this procedure.

Paradigm: Write $3 \cos(3x) - \sin(3x)$ in the form $A \cos(\omega x + \phi)$

STEP 1: Write the function as the \Re part of a complex number times a complex exponential.

The trick here is to make the real part of the complex number the coefficient of the cos term and the imaginary part of the complex number the *negative* of the coefficient of the sin term. $\cos(3x) = \Re[e^{i3x}]$ and $\sin(3x) = \Re[-ie^{i3x}]$ so $3 \cos(3x) - \sin(3x) = \Re[3e^{i3x} - (-ie^{i3x})] = \Re[(3 + i)e^{i3x}]$

STEP 2: Write the coefficient of the complex expression in polar form.

This is just the conversion to polar coordinates. $|3 + i| = \sqrt{3^2 + 1^2} = \sqrt{10}$. $\arg(3 + i) = \arctan(1/3) \approx .322$. So $3 + i \approx \sqrt{10}e^{.322i}$. (Don't forget in the exercises that you sometimes have to adjust the arctan by $\pm\pi$ to get the correct arg.)

STEP 3: Multiply out the coefficient in polar form with the complex exponential.

$$(3 + i)e^{i3x} \approx \sqrt{10}e^{.322i}e^{i3x} = \sqrt{10}e^{i(3x+.322)}$$

STEP 4: Take \Re part.

$$3 \cos(3x) - \sin(3x) = \Re[(3 + i)e^{i3x}] \approx \Re[\sqrt{10}e^{i(3x+.322)}] = \sqrt{10} \cos(3x + .322).$$

Of course, you can go the other way too. People don't normally worry as much about this because the trig identities used in changing expressions of the form $A \cos(\omega x - \phi)$ to $a \cos(\omega x) + b \sin(\omega x)$ are just the addition of angle formulas and those are more familiar to most people.

Example: Write $2 \cos(2.7x - 1.3)$ in the form $a \cos(2.7x) + b \sin(2.7x)$.

One way is just to reverse the process of the paradigm as follows

$$\begin{aligned}
 2 \cos(2.7x - 1.3) &= \Re(2e^{(2.7x-1.3)i}) \\
 &= \Re(2e^{-1.3i} e^{2.7ix}) \\
 &= \Re((2 \cos(-1.3) + 2i \sin(-1.3))(\cos(2.7x) + i \sin(2.7x))) \\
 &\approx \Re((0.5350 - 1.9271i)(\cos(2.7x) + i \sin(2.7x))) \\
 &= \Re(0.5350 \cos(2.7x) \\
 &\quad + 1.9271 \sin(2.7x) + 0.5350i \sin(2.7x) - 1.9271i \cos(2.7x)) \\
 &= 0.5350 \cos(2.7x) + 1.9271 \sin(2.7x)
 \end{aligned}$$

Another way is to use the trig identities

$$\begin{aligned}
 \cos(\theta \pm \phi) &= \cos(\theta) \cos(\phi) \mp \sin(\theta) \sin(\phi) \\
 \sin(\theta \pm \phi) &= \sin(\theta) \cos(\phi) \pm \cos(\theta) \sin(\phi)
 \end{aligned}$$

which gives the following

$$\begin{aligned}
 2 \cos(2.7x - 1.3) &= 2 \cos(2.7x) \cos(1.3) + 2 \sin(2.7x) \sin(1.3) \\
 &\approx 0.5350 \cos(2.7x) + 1.9271 \sin(2.7x)
 \end{aligned}$$

As you can see, the trig identities work out quicker this time, as long as you keep straight where you have \pm and where you have \mp in the identities.

Exercises: Write the following expressions in the form $A \cos(\omega x - \phi)$.

- | | |
|------------------------------------|----------------------------------|
| (1) $\cos(x) + \sin(x)$ | (2) $\cos(3x) - 4 \sin(3x)$ |
| (3) $2 \cos(2x) - \sin(2x)$ | (4) $\sqrt{3} \cos(x) - \sin(x)$ |
| (5) $\cos(4x) + \sqrt{3} \sin(4x)$ | (6) $3 \cos(2x) - 2 \sin(2x)$ |
| (7) $\sin(x) + 4 \cos(x)$ | (8) $4 \sin(3x) + 3 \cos(3x)$ |
| (9) $12 \sin(2x) + 5 \cos(2x)$ | (10) $\sin(5x) + \cos(5x)$ |

Write the following expressions in the form $a \cos(\omega x) + b \sin(\omega x)$

- | | |
|---------------------------|--------------------------|
| (11) $\cos(x - \pi/3)$ | (12) $\cos(x + \pi/4)$ |
| (13) $3 \sin(2x + \pi/6)$ | (14) $4 \sin(x - \pi/4)$ |
| (15) $2 \cos(3x - 1)$ | (16) $-3 \sin(5x - .3)$ |

Graphing a complex function, like $w = e^z$, presents a problem. Each complex variable requires two real dimensions to plot. So if we try to plot w versus z , we need 2 dimensions for z and 2 dimensions for w , a total of 4 dimensions. Unfortunately, I don't have any 4-dimensional graph paper. But with a little cleverness, we can still find a way to graph 4 dimensions at once. The key observation is that each dimension just represents something that can vary. Normally when we draw a graph what we vary is position. But we can also vary other things. In this lab we will graph complex functions where we vary both position and color. In particular, the base plane will be the complex plane of the independent variable, z . The height of the point above the base plane will be the modulus of the dependent variable $w = f(z)$. The color of the point will correspond to the argument of the dependent variable $w = f(z)$. After trying to write the description of how this works, I've discovered this isn't something that can be done with just plain text. So please go sit down at a computer somewhere and we'll continue this discussion. Note that this lab looks better on a computer with at least a 256 color display.

There, I hope you're comfortable and ready to begin. Get into matlab and give the command `xc6`. This will create two windows (and it will take a little time to get them all finished). The first window is a smaller one called Reference. It shows how the colors correspond to angles. Cyan (light blue) is the positive Real axis, red is the negative Real axis, dark blue shading to magenta is the positive imaginary axis and green is the negative imaginary axis. Initially it displays the plot of the function $w = z$. The height of the plot is the modulus of w , and since $|w| = |z|$, the height is equal to the distance from the origin and the surface is a cone. The colors as you look at the cone match the colors of the reference window, since $\arg(w) = \arg(z)$. Now change the function to $w = z^2$. Remember that you have to use `^` with a period in front of it, or you won't get the right graph. It takes some time for the new graph to be computed and plotted. Both the shape and colors have changed. Since $|w| = |z^2| = |z|^2$, the height is the distance from the origin squared and the graph has the shape of a paraboloid. To get a better look at the colors, click on the **Top View** button to see the view staring directly down from the top. In this view you can't see the shape any more but you can see all of the colors, none of them are obscured by having part of the graph covering up another part. You should note that both the positive and negative real axes are colored cyan. That makes sense, the square of a real number is a positive real number, no matter if the original real number is positive or negative, and cyan is the color of positive reals. Both the positive and negative imaginary axes are red. That also checks; the square of a purely imaginary number should be a negative real number and red is the color of negative reals. Note that the colors go all the way around

the cycle twice now. If we write $z = re^{i\theta}$, then $w = z^2 = r^2e^{i2\theta}$ and so $\arg(w) = 2\theta$ and as θ varies from 0 to 2π , 2θ varies from 0 to 4π , so the colors should complete two full cycles. Also note that the colors change smoothly and continuously, with one exception. All the different colors come together at the origin. What is special about the origin? The origin is the only root of the polynomial z^2 . Finally note that the colors keep the same order as in the reference window: blue, green, yellow, red, purple as you move clockwise.

Next change the function to $w = z^{-1}$ (you will have to use $z.^{-1}$, $1./z$ won't work). If you use the side view, the graph looks like a tall peak on a flat plane. Well $1/0 = \infty$ (sometimes) so the graph should blow up at the origin. If you look at the top view, it might make you think the colors are the same as in the reference window. They go all around the cycle once and they all come together at the origin. But they go backwards: blue, purple, red, yellow, green as you move clockwise. Hmm, let's see. If we write $z = re^{i\theta}$ then $w = z^{-1} = r^{-1}e^{-i\theta}$ and $\arg(w) = -\theta$ so as θ runs from 0 to 2π , $\arg(w)$ runs from 0 to -2π . So the argument should indeed reverse the colors.

You have several problems where you are asked to look at graphs of rational functions. You should be able to distinguish the zeros and poles (infinities) of the functions just by looking at the colors and seeing where they come together and how they spin. You will also have two final questions dealing with the exponential function and the cosine rather than rational functions, but fortunately they should be pretty easy.

- (1) Consider the function $w = z \times (z + 1)^2$. Describe how the colors look around the roots of this polynomial. How can you distinguish the double root at $z = -1$ from the single root at $z = 0$ from the colors?
- (2) Consider the rational function $w = z^2/(z^2 + 1)$. Describe how the colors look around the roots and poles of this function. How can you distinguish the poles at $z = \pm i$ from the root at $z = 0$?
- (3) Note how the colors look as you go around the outside edge of the previous plot, $w = z^2/(z^2 + 1)$. Next look at how the colors look as you go around the outside edge of the plot of $w = z^3/(z^2 + 1)$ and $w = z/(z^2 + 1)$. Can you find a pattern about the number of times the colors cycle around the edge and the number of poles and zeros inside?
- (4) In §2 you learned that the exponential function was periodic with imaginary period. Consider the graph of the function $w = \exp(2\pi z)$. How can you see the periodicity in this graph?
- (5) While the exponential function is periodic with imaginary period, the cosine function is periodic and bounded with real period. But how does it behave as you move in the

imaginary direction? Is the cosine function still bounded if you treat it as a function of a complex variable?

§5 THEORETICAL CONSIDERATIONS

Discussion: We are now going to consider higher order equations such as $y'' - y = 0$. While the techniques we use will generally work for any order equation, we will concentrate mainly on second order equations as they are the most common type in applications. The prominence of second order equations is a consequence of the fact that Newton's second law relates force to acceleration, the second derivative of position with respect to time. It may seem like the typical perversity of a math professor to start with theoretical considerations rather than solving problems, but I have my reasons. We will need to know what the general solution to a higher order equation looks like, however, before we can start searching for it. Accordingly we will start by discussing the theory of linear higher order equations, which will give us a clue about what we are looking for.

One way to solve higher order equations is by reducing them to collections of first order equations. Unfortunately, this is easy only if the equation is linear and constant coefficient. On the other hand, linear constant coefficient equations are very common in applications so this defect isn't as troubling as it might first seem.

A differential equation is **linear** if it can be written in the form

$$a_n(x) \frac{d^n y}{dx^n} + \cdots + a_1(x) \frac{dy}{dx} + a_0(x)y = f(x)$$

For linear differential equations we have the following theorem, which we shall not prove.

THEOREM. *If $a_{n-1}(x), \dots, a_1(x), a_0(x), f(x)$ are all continuously differentiable in an interval about x_0 , then the initial value problem*

$$\begin{aligned} \frac{d^n y}{dx^n} + a_{n-1}(x) \frac{d^{n-1} y}{dx^{n-1}} + \cdots + a_1(x) \frac{dy}{dx} + a_0(x)y &= f(x) \\ y(x_0) = y_0, \quad y'(x_0) = y_1, \dots, y^{(n-1)}(x_0) &= y_{n-1} \end{aligned}$$

has a unique solution in some interval about x_0 .

Note that the conditions for the initial value problem in the theorem are the values of the function and its derivatives at a single point. While it is possible to consider other sorts

of conditions, those other sorts of problems (prominently “boundary value problems”) do not necessarily have solutions, and if they have solutions they aren’t necessarily unique. In this class, we will concentrate on initial value problems. Fortunately in most physical situations, these are also the natural conditions to know for a problem.

While we are interested in understanding solutions of linear differential equations, much of the information we need holds for general linear equations. It will be convenient to introduce the general notion of a linear operator in what follows.

An **operator** is a mapping that takes functions to functions. An operator, L , is **linear** if

$$L(f + g) = Lf + Lg \quad \text{and} \quad L(cf) = cL(f)$$

for all functions f and g and all constants c .

Examples:

Let D denote the differentiation operator. This is an operator because the derivative of a function is another function (though possibly the constant function or the zero function). This is also a linear operator since

$$D(f + g) = Df + Dg \quad \text{and} \quad Dcf = cDf$$

by the usual rules of differentiation.

A more complicated example is

$$Ly = \frac{d^n y}{dx^n} + a_{n-1}(x) \frac{d^{n-1} y}{dx^{n-1}} + \cdots + a_1(x) \frac{dy}{dx} + a_0(x)y$$

This is also a linear operator because

$$\begin{aligned} L(y + z) &= \frac{d^n(y + z)}{dx^n} + a_{n-1}(x) \frac{d^{n-1}(y + z)}{dx^{n-1}} + \cdots + a_1(x) \frac{d(y + z)}{dx} + a_0(x)(y + z) \\ &= \frac{d^n y}{dx^n} + \frac{d^n z}{dx^n} + \cdots + a_0(x)y + a_0(x)z \\ &= \left(\frac{d^n y}{dx^n} + \cdots + a_0(x)y \right) + \left(\frac{d^n z}{dx^n} + \cdots + a_0(x)z \right) \\ &= Ly + Lz \end{aligned}$$

since the derivative of the sum is the sum of the derivatives and also

$$\begin{aligned} L(cy) &= \frac{d^n(cy)}{dx^n} + a_{n-1}(x) \frac{d^{n-1}(cy)}{dx^{n-1}} + \cdots + a_1(x) \frac{d(cy)}{dx} + a_0(x)(cy) \\ &= c \frac{d^n y}{dx^n} + ca_{n-1}(x) \frac{d^{n-1}y}{dx^{n-1}} + \cdots + ca_1(x) \frac{dy}{dx} + ca_0(x)y \\ &= c \left(\frac{d^n y}{dx^n} + a_{n-1}(x) \frac{d^{n-1}y}{dx^{n-1}} + \cdots + a_1(x) \frac{dy}{dx} + a_0(x)y \right) \\ &= cLy \end{aligned}$$

since the derivative of a constant times a function is the constant times the derivative of the function.

This last example shows that a linear differential equation can be written as

$$Ly = f$$

where L is a linear operator. We call any operator of the form L a **linear differential operator**. A linear differential equation is **homogeneous** if it can be written in the form

$$Ly = 0.$$

A linear differential equation of the form

$$Ly = f$$

where $f(x)$ is not identically 0 is called **inhomogeneous**.

THEOREM. Suppose L is a linear operator and $Ly = 0$ and $Lz = 0$. Then $L(c_1y + c_2z) = 0$.

PROOF: $L(c_1y + c_2z) = L(c_1y) + L(c_2z) = c_1Ly + c_2Lz = 0 + 0 = 0$

So if we have two solutions to a linear homogeneous equation, all their linear combinations are also solutions. This is true for all linear equations, including linear differential equations of course. For inhomogeneous equations, the sum of two solutions will not be a solution but we do have the following.

THEOREM. Suppose L is a linear operator and $Ly = 0$ and $Ly_p = f$. Then $L(y + y_p) = f$.

PROOF: $L(y + y_p) = Ly + Ly_p = 0 + f = f$.

So if we have a solution to a linear inhomogeneous equation and a solution to the corresponding linear homogeneous equation their sum is another solution to the linear inhomogeneous equation. What is more, *every* solution to the linear inhomogeneous equation can be found from any *single* solution to the linear inhomogeneous equation in this fashion.

THEOREM. Suppose y_p is a solution to $Ly_p = f$ where L is a linear operator. Then the set

$$\{y + y_p : Ly = 0\}$$

is the set of all solutions to $Lz = f$.

PROOF: Suppose $Lz = f$. Then $L(z - y_p) = Lz - Ly_p = f - f = 0$. So $z = (z - y_p) + y_p$ with $L(z - y_p) = 0$ and so z is in the specified set.

So to find the general solution of a linear inhomogeneous equation it is enough to find one particular solution and then find all the solutions to the corresponding homogeneous equation. Now just as for first order equations, we expect the general solution to a higher order equation to involve arbitrary constants. Since the initial value problem for an n^{th} order equation has a unique solution when given n conditions, we should expect the general solution to an n^{th} order equation to involve n arbitrary constants. Furthermore, given n different solutions of a homogeneous equation, $y_1(x), \dots, y_n(x)$, we have a theorem that assures us that $c_1y_1(x) + \dots + c_ny_n(x)$ is also a solution to the homogeneous equation. This solution appears to involve n arbitrary constants, but looks can be deceiving. Consider the following example.

Example: $y'' + 4y' + 3y = 0$

Two different solutions are $y_1(x) = e^{-x}$ and $y_2(x) = 2e^{-x}$. So $y(x) = c_1e^{-x} + c_2(2e^{-x})$ is also a solution to the equation and it involves two arbitrary constants c_1 and c_2 . But e^{-3x} is also a solution and it can not be obtained by any choice of c_1 and c_2 . So while $y(x)$ is always a solution of the equation for any choice of c_1 and c_2 , it doesn't give all the solutions, despite its two arbitrary constants.

The difficulty here, of course, is that we really only have one constant split into two pieces. $y(x) = (c_1 + 2c_2)e^{-x}$ and the one arbitrary constant is $c_1 + 2c_2$. There was nothing wrong with our general approach, the problem was that our two different solutions just weren't different enough. We need to replace the term different by the technical condition, linearly independent. A collection of n functions $y_1(x), \dots, y_n(x)$ is **linearly independent** if the only choice of constants c_1, \dots, c_n for which

$$c_1y_1(x) + \dots + c_ny_n(x) = 0$$

for all x is $c_1 = 0, \dots, c_n = 0$.

Chapter 2: Linear Constant Coefficient Higher Order Equations

THEOREM. If $y_1(x), \dots, y_n(x)$ are linearly independent solutions to an n^{th} order linear homogeneous differential equation $Ly = 0$, then the general solution to $Ly = 0$ is $y(x) = c_1y_1(x) + \dots + c_ny_n(x)$.

Of course checking linear independence by using the definition is very messy. There is a simple test for linear independence using the Wronskian of the set of functions. The **Wronskian** of a pair of functions $y_1(x)$ and $y_2(x)$ is $W(y_1, y_2) = y_1(x)y_2'(x) - y_2(x)y_1'(x)$. This definition only works for pairs of functions. The definition can be extended to a collection of n functions, but it depends on the concept of a determinant of an $n \times n$ matrix. If you have seen determinants before, stop by my office and we will go over Wronskians in the general case. If you haven't seen determinants, then you ought to take Math 551 and make their acquaintance.

THEOREM. If $W(y_1, y_2) \neq 0$ for some value of x , then y_1 and y_2 are linearly independent.

PROOF: Suppose

$$c_1y_1(x) + c_2y_2(x) = 0$$

for all x . Differentiating this equation we obtain

$$c_1y_1'(x) + c_2y_2'(x) = 0$$

for all x as well. Now if we multiply the second equation by $y_1(x)$ and the first equation by $y_1'(x)$ and subtract we obtain

$$c_2[y_1(x)y_2'(x) - y_2(x)y_1'(x)] = 0$$

or $c_2W(y_1, y_2)(x) = 0$ for all x by the definition of the Wronskian. Since $W(y_1, y_2)(x) \neq 0$ for some x , we must have $c_2 = 0$. On the other hand, if we multiply the first equation by $y_2'(x)$ and the second equation by $y_2(x)$ and subtract we obtain $c_1W(y_1, y_2)(x) = 0$ for all x . Again, since $W(y_1, y_2)(x) \neq 0$ for some x , it must also be the case that $c_1 = 0$. But then we have shown that the only choice of constants c_1 and c_2 for which $c_1y_1(x) + c_2y_2(x) = 0$ is $c_1 = c_2 = 0$ and so y_1 and y_2 are linearly independent.

Note that the theorem just says that if the Wronskian is non-zero then the functions are linearly independent. It doesn't say that if the functions are linearly independent then the Wronskian is non-zero. It doesn't say that because it isn't true, but we will not

have occasion to worry about situations where linearly independent functions have a zero Wronskian in this course.

§6 HOMOGENEOUS EQUATIONS

Discussion: As the results in the first section showed, solving the homogeneous equation plays a special role in solving the general problem. One way to solve such problems is to factor the equation. Consider the example

$$y'' - y = 0$$

We factor the equation to a pair of first order equations. Let $D = d/dx$. Then $D^2 = DD = (d/dx)(d/dx) = d^2/dx^2$. Write the equation $y'' - y = 0$ as

$$\begin{aligned} (D^2 - 1)y &= 0 \\ (D - 1)(D + 1)y &= 0 \\ (D - 1)y = 0 \quad \text{OR} \quad (D + 1)y &= 0 \\ y = Ce^x \quad \text{OR} \quad y = Ke^{-x} & \\ \text{General Solution: } y &= Ce^x + Ke^{-x} \end{aligned}$$

where we use the fact from the previous section that the sum of solutions to a linear homogeneous equation is also a solution. We can then check this solution by computing

$$\begin{aligned} y &= Ce^x + Ke^{-x} \\ y' &= Ce^x - Ke^{-x} \\ y'' &= Ce^x + Ke^{-x} \end{aligned}$$

and so $y'' - y = 0$ as desired. So we have solved a linear constant coefficient homogeneous equation by treating it as a polynomial with a “variable” D . If the equation was not constant coefficient, the method would become much harder because we wouldn’t be able to factor in the same fashion we are used to. This is because the algebra of differential operators is different from the usual algebra learned in high school. For example

$$\begin{aligned} (D + x)(D + 1)y &= y'' + (x + 1)y' + xy \\ (D + 1)(D + x)y &= y'' + (x + 1)y' + (x + 1)y. \end{aligned}$$

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So we see $(D + x)(D + 1)$ is not equal to $(D + 1)(D + x)$. In other words, the commutative law fails for differential operators. Differential Algebra is a very interesting subject but it quickly gets very difficult. That is why we restrict our attention here to the constant coefficient case. For the constant coefficient case everything works the same as in high school algebra and let's be thankful for it. There are three points to remember in solving these equations by factoring:

- 1) The solution to $(D - r)y = 0$ is $y = Ce^{rx}$.
- 2) The solution to $(D - r)^2y = 0$ is $y = Cxe^{rx} + Ke^{rx}$.
- 3) The solution to $(D - r)^ny = 0$ is $y = C_{n-1}x^{n-1}e^{rx} + C_{n-2}x^{n-2}e^{rx} + \dots + C_0e^{rx}$.

In all three cases r represents a constant. The first equation is both separable and linear and can be easily solved. The second equation is a little trickier. We solve it by converting to a system. Let $u = (D - r)y$. Then the equation is $(D - r)u = 0$ and so $u = Ce^{rx}$. Now we solve $Ce^{rx} = (D - r)y$ (which is a first order linear equation) for y to obtain $y = Cxe^{rx} + Ke^{rx}$ as advertised. Repeating this process we get the solution to the general case in the third equation.

Paradigm: Find the general solution of $y'' + 3y' - 10y = 0$

STEP 1: Write in operator form (with D for d/dx)

$$(D^2 + 3D - 10)y = 0$$

STEP 2: Factor and find roots.

$$(D - 2)(D + 5)y = 0 \text{ so the roots are } 2, -5.$$

STEP 3: Write the general solution.

$$y(x) = Ce^{2x} + Ke^{-5x}$$

EXAMPLE: $y'' + 4y' + 3y = 0$, $y(0) = 1$, $y'(0) = 1$

FIRST: Find the general solution.

Step 1: $(D^2 + 4D + 3)y = 0$

Step 2: $(D + 3)(D + 1)y = 0$ so the roots are $-3, -1$

Step 3: $y = Ce^{-3x} + Ke^{-x}$

SECOND: Plug in the initial values and solve for the constants.

$$\begin{aligned} y(0) &= Ce^{-3 \times 0} + Ke^{-1 \times 0} = C + K \stackrel{\text{set}}{=} 1 \\ y'(x) &= -3Ce^{-3x} - Ke^{-x} \\ y'(0) &= -3Ce^{-3 \times 0} - Ke^{-1 \times 0} = -3C - K \stackrel{\text{set}}{=} 1 \end{aligned}$$

Taking 3 times the first equation and adding it to the second we find $2K = 4$ so $K = 2$ and $C = -1$. This gives us the solution

$$y = -e^{-3x} + 2e^{-x}$$

EXAMPLE: $y'' + 10y' + 25y = 0$

Step 1: $(D^2 + 10D + 25)y = 0$

Step 2: $(D + 5)^2y = 0$, double root of -5

Step 3: $y = Cxe^{-5x} + Ke^{-5x}$

Now we consider the equation $y'' + y = 0$. If we try to solve this equation using our new techniques, we obtain the operator $D^2 + 1$ which has roots $\pm i$, so the general solution is $c_1e^{ix} + c_2e^{-ix}$. This is in fact the correct general *complex* solution, provided we allow c_1 and c_2 to be arbitrary complex constants. But since we started with a real problem we probably want a real answer. One way to deal with that is to obtain the general complex solution as above and find all the complex solutions which are actually real solutions. The algebra involved in that is rather messy though. Fortunately, there is a quicker way that will give us two linearly independent real solutions from a single complex solution.

THEOREM. *If $e^{(\alpha+\beta i)x}$ is a solution of the equation $ay'' + by' + cy = 0$ where a, b and c are all real, then $e^{\alpha x} \cos(\beta x)$ and $e^{\alpha x} \sin(\beta x)$ are two linearly independent solutions of $ay'' + by' + cy = 0$.*

PROOF: We are given that

$$a(e^{(\alpha+\beta i)x})'' + b(e^{(\alpha+\beta i)x})' + c(e^{(\alpha+\beta i)x}) = 0$$

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Since $\Re[y'(x)] = (\Re[y(x)])'$ and $\Im[y'(x)] = (\Im[y(x)])'$, it follows that

$$\begin{aligned} a(\Re[e^{(\alpha+\beta i)x}])'' + b(\Re[e^{(\alpha+\beta i)x}])' + c(\Re[e^{(\alpha+\beta i)x}]) &= 0 \\ a(\Im[e^{(\alpha+\beta i)x}])'' + b(\Im[e^{(\alpha+\beta i)x}])' + c(\Im[e^{(\alpha+\beta i)x}]) &= 0 \end{aligned}$$

So $e^{\alpha x} \cos(\beta x)$ and $e^{\alpha x} \sin(\beta x)$ are both real solutions to the given equation. Linear independence can be quickly checked using the Wronskian.

Using this theorem we can now handle complex roots efficiently. One point that often bothers students at this stage is that we only need one complex solution to find the complete general solution. What happens to the other complex solution? It is a theorem that complex roots to a real polynomial always come in complex conjugate pairs, $\alpha \pm \beta i$. So since the operator has real coefficients, if $\alpha + \beta i$ is one root of the polynomial equation, $\alpha - \beta i$ must be another root of the equation, and $\alpha - \beta i$ will give rise to the same pair of real roots. So we really get our pair of real roots from a pair of complex roots.

Another point to be noted is that we can write the pair of real solutions we get from our complex conjugate pair of roots $\alpha \pm \beta i$ as either $C_1 e^{\alpha x} \cos(\beta x) + C_2 e^{\alpha x} \sin(\beta x)$ or as $A e^{\alpha x} \cos(\beta x - \phi)$ (with arbitrary constants A and ϕ) using the techniques of section 3. The first form is usually the easiest for solving initial value problems while the second form is usually the easiest for sketching solutions. And, of course, in initial value problems where you have to sketch solutions, you may want to solve first in one form to solve the initial value problem and then switch to the second form to draw your sketch.

EXAMPLE: $y'' + 2y' + 2y = 0$

Step 1: Write the equation in operator form

$$(D^2 + 2D + 2)y = 0$$

Step 2: Find the roots of the equation

roots are $-1 \pm i$

Step 3: The general solution is $A e^{-x} \cos(x - \phi)$ (or $C_1 e^{-x} \cos(x) + C_2 e^{-x} \sin(x)$ in the alternate form).

EXAMPLE: $y'' + y = 0$, $y(0) = 1$, $y'(0) = 1$

FIRST: Find the general solution.

Step 1: $(D^2 + 1)y = 0$

Step 2: roots are $\pm i$

Step 3: $y(x) = C \cos(x) + K \sin(x)$ (the easiest form for initial value problems).

SECOND: Plug in the initial values and solve for the constants

$$\begin{aligned} y(0) &= C \stackrel{\text{set}}{=} 1 \\ y'(x) &= -C \sin(x) + K \cos(x) \\ y'(0) &= K \stackrel{\text{set}}{=} 1 \end{aligned}$$

So the solution to the initial value problem is $y(x) = \cos(x) + \sin(x)$.

EXAMPLE: $y''' + y = 0$.

Step 1: $(D^3 + 1)y = 0$

Step 2: $D^3 + 1 = (D + 1)(D^2 - D + 1)$ so the roots are -1 and $1/2 \pm (\sqrt{3}/2)i$.

Step 3: The solution corresponding to the root -1 is e^{-x} . The solutions corresponding to the complex conjugate pair $1/2 \pm \sqrt{3}/2$ are $Ae^{x/2} \cos(\sqrt{3}x/2 - \phi)$. So the general solution is

$$y(x) = C_1 e^{-x} + Ae^{x/2} \cos(\sqrt{3}x/2 - \phi).$$

EXAMPLE: $y'''' + 8y''' + 27y'' + 38y' + 26y = 0$

Step 1: $(D^4 + 8D^3 + 27D^2 + 38D + 26)y = 0$

Step 2: $D^4 + 8D^3 + 27D^2 + 38D + 26 = (D^2 + 2D + 2)(D^2 + 6D + 13) = 0$ so the roots are $-1 \pm i$ and $-3 \pm 2i$.

Step 3: $y = Ae^{-x} \cos(x - \phi) + Be^{-3x} \cos(2x - \psi)$, or written in the alternate form $y = C_1 e^{-x} \cos(x) + C_2 e^{-x} \sin(x) + C_3 e^{-3x} \cos(2x) + C_4 e^{-3x} \sin(2x)$.

EXAMPLE: $y'''' + 4y''' + 24y'' + 40y' + 100y = 0$.

Step 1: $(D^4 + 4D^3 + 24D^2 + 40D + 100)y = 0$

Step 2: $(D^2 + 2D + 10)^2 = 0$ so we have *double* roots of $-1 \pm 3i$.

Step 3: We handled double real roots by using both the usual solution and the solution multiplied by x . We do exactly the same thing with double complex roots. So the solution is $y = Ae^{-x} \cos(3x - \phi) + Bxe^{-x} \cos(3x - \psi)$ or in the alternate form $y = C_1e^{-x} \cos(3x) + C_2e^{-x} \sin(3x) + C_3xe^{-x} \cos(3x) + C_4xe^{-x} \sin(3x)$.

Exercises:

- | | |
|----------------------------------|---|
| (1) $y'' + 4y = 0$ | (2) $y'' + 2y' + 5y = 0$ |
| (3) $y'' + 6y' + 10y = 0$ | (4) $y'' - 2y' + 2y = 0$ |
| (5) $y'' + 5y' + 10y = 0$ | (6) $y''' + y'' - y' - y = 0$ |
| (7) $y'' - 2y' + 5y = 0$ | (8) $y'' + 4y' + 13y = 0$ |
| (9) $y''' + 9y' = 0$ | (10) $y''' + 5y'' + y' - 7y = 0$ |
| (11) $y'' - y = 0$ | (12) $y'' + 2y' = 0$ |
| (13) $y'' + 5y' + 4y = 0$ | (14) $y'' + 2y' + y = 0$ |
| (15) $y''' + y'' - 6y' = 0$ | (16) $y'''' - 4y'' + 4y = 0$ |
| (17) $y'' + 3y' - 3y = 0$ | (18) $y'' - 2y' - 5y = 0$ |
| (19) $y''' - 3y'' + 3y' - y = 0$ | (20) $y'''' - 4y''' + 6y'' - 4y' + y = 0$ |
-
- | | | | |
|---------------------------------|-----------------|-----------------|--------------|
| (21) $y'' + 4y' + 5y = 0,$ | $y(0) = 1,$ | $y'(0) = 0$ | |
| (22) $y'' - 9y' + 10y = 0,$ | $y(0) = 1,$ | $y'(0) = 0$ | |
| (23) $y'' + 4y' + 8y = 0,$ | $y(\pi) = 1,$ | $y'(\pi) = 1$ | |
| (24) $y'' + 10y' + 29y = 0,$ | $y(0) = 1,$ | $y'(0) = 1$ | |
| (25) $y'' + 2y' + 10y = 0,$ | $y(0) = 1,$ | $y'(0) = 0$ | |
| (26) $y'' - 4y' + 20y = 0,$ | $y(0) = 0,$ | $y'(0) = 1$ | |
| (27) $y'' + 9y = 0,$ | $y(\pi) = 1,$ | $y'(\pi) = 2$ | |
| (28) $y'' + 6y' + 25y = 0,$ | $y(\pi/2) = 2,$ | $y'(\pi/2) = 1$ | |
| (29) $y''' + y'' + y' + y = 0,$ | $y(0) = 1,$ | $y'(0) = 0,$ | $y''(0) = 0$ |
| (30) $y''' + y'' - 2y = 0,$ | $y(0) = 0,$ | $y'(0) = 1,$ | $y''(0) = 0$ |

- (31) $y'' + 2y' - 3y = 0, \quad y(0) = 1, \quad y'(0) = 0$
 (32) $y'' - 3y' + 2y = 0, \quad y(0) = 0, \quad y'(0) = 1$
 (33) $y'' - 2y' - 8y = 0, \quad y(2) = 1, \quad y'(2) = 0$
 (34) $y''' + 2y'' - 5y' - 6y = 0, \quad y(0) = 1, \quad y'(0) = 0, \quad y''(0) = 0$
 (35) $y'' - 4y' - 21y = 0, \quad y(1) = 0, \quad y'(1) = 1$
 (36) $y'' + y' - \frac{3}{4}y = 0, \quad y(1) = 1, \quad y'(1) = 0$
 (37) $y'' + 2y' + y = 0, \quad y(0) = 1, \quad y'(0) = 2$
 (38) $y''' + 6y'' + 9y' = 0, \quad y(0) = 1, \quad y'(0) = 0, \quad y''(0) = 0$
 (39) $y''' + 4y'' + 3y' = 0, \quad y(0) = 3, \quad y'(0) = -3, \quad y''(0) = 11$
 (40) $y'''' - 8y'' + 16y = 0, \quad y(0) = 0, \quad y'(0) = 0, \quad y''(0) = 0, \quad y'''(0) = 0$
 (41) $y''' + y'' + y' + y = 0$
 (42) $y''' - y'' + y' - y = 0$
 (43) $y''' + 4y'' + 6y' + 4y = 0$
 (44) $y'''' + 8y'' + 16y = 0$
 (45) $y''' + 5y'' + 4y' - 10y = 0, \quad y(0) = 1, \quad y'(0) = 0, \quad y''(0) = 0$