§4 Exact Equations

Discussion: The general solution to a first order equation has 1 arbitrary constant. If we solve for that constant, we can write the general solution to a first order equation in the form

$$F(x, y) = K$$

where F is some function which depends upon the equation and K is the constant. From this solution, we can now try to recover the original equation. Differentiate both sides with respect to x and use the chain rule for functions of several variables to obtain

$$\frac{\partial F}{\partial x} + \frac{\partial F}{\partial y}\frac{dy}{dx} = 0$$

This now suggests a technique for solving first order differential equations; find a function F so that the equation takes on the form above and then the solution is $F(x, y) = K$. If we start with a general equation

$$M(x, y) + N(x, y)\frac{dy}{dx} = 0$$

we then have the question of how to find F. In fact, first we should check that such an F exists. Comparing this equation with the previous equation, we see that we must have

$$\frac{\partial F}{\partial x} = M(x, y)$$
$$\frac{\partial F}{\partial y} = N(x, y)$$

That M and N are the two different partials of a single function means that $[M, N]$ forms a gradient field in the terminology of Calculus III. And, of course, all of you remember that a necessary and sufficient condition that a vector field is locally a gradient field is that the curl of the vector field is 0. In this setup, that means there is a function F satisfying the pair of equations above if and only if

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Once we have checked that an F exists, it is usually a straightforward process to find F by integration. The process is sufficiently lengthy though that it is worthwhile to check that an F exists before trying to construct it.

Example:

$$2xy + (x^2 + 1)\frac{dy}{dx} = 0$$

We check that

$$\frac{\partial(2xy)}{\partial y} = 2x = \frac{\partial(x^2 + 1)}{\partial x}$$

so the equation is exact. So we have

$$\begin{aligned}\frac{\partial F}{\partial x} &= 2xy \\ \frac{\partial F}{\partial y} &= x^2 + 1\end{aligned}$$

We now integrate the first partial derivative to obtain

$$\begin{aligned}\int \partial F &= \int 2xy\,\partial x \\ F(x, y) &= x^2 y + C(y)\end{aligned}$$

This requires a bit of explanation. We do a partial integration because we are undoing a partial differentiation. So we integrate with respect to x and treat y as a constant. Finally, we end up with an arbitrary function of y, $C(y)$, instead of the usual arbitrary constant of integration since we are treating y as a constant. So from our condition that $\partial F/\partial x = 2xy$ we obtain a formula for F. But we can obtain another formula from $\partial F/\partial y = x^2 + 1$.

$$\begin{aligned}\int \partial F &= \int x^2 + 1\,\partial y \\ F(x, y) &= x^2 y + y + \tilde{C}(x)\end{aligned}$$

where $\tilde{C}(x)$ is an arbitrary function of x. We now have two different definitions of F from our two different equations.

$$\begin{aligned}F(x, y) &= x^2 y + C(y) \\ F(x, y) &= x^2 y + y + \tilde{C}(x)\end{aligned}$$

Comparing our two formulas, we see that they agree if we set $C(y) = y$ and $\tilde{C}(x) = 0$. So we end up with $F(x, y) = x^2 y + y$ and our solution is

$$x^2 y + y = K$$

where K is an arbitrary constant.

We are now ready to give the paradigm.

Paradigm: $(x^2 + y) + (x + \cos(y))\dfrac{dy}{dx} = 0$

STEP 1: Test for exactness

$$\frac{\partial(x^2 + y)}{\partial y} = 1 = \frac{\partial(x + \cos(y))}{\partial x}$$

so the equation is exact.

STEP 2: Write the partial differential equations.

$$\frac{\partial F}{\partial x} = x^2 + y$$
$$\frac{\partial F}{\partial y} = x + \cos(y)$$

STEP 3: Integrate the first partial differential equation.

$$\int \partial F = \int x^2 + y \, \partial x$$
$$F(x, y) = x^3/3 + xy + C(y)$$

STEP 4: Integrate the second partial differential equation.

$$\int \partial F = \int x + \cos(y) \, \partial y$$
$$F(x, y) = xy + \sin(y) + \tilde{C}(x)$$

STEP 5: Equate expressions for $F(x, y)$.

$$F(x, y) = x^3/3 + xy + C(y)$$
$$F(x, y) = xy + \sin(y) + \tilde{C}(x)$$

So $C(y) = \sin(y)$ and $\tilde{C}(x) = x^3/3$ and

$$F(x, y) = x^3/3 + xy + \sin(y)$$

STEP 6: Solution is $F(x, y) = K$.

$$x^3/3 + xy + \sin(y) = K$$

EXAMPLE: $x^2 \dfrac{dy}{dx} = x^3 - 2xy, \qquad y(1) = 3$

FIRST: Find the general solution.

Step 1: Rewriting this as $(2xy - x^3) + x^2 \dfrac{dy}{dx} = 0$ the test for exactness is

$$\frac{\partial(2xy - x^3)}{\partial y} = 2x = \frac{\partial x^2}{\partial x}$$

so the equation is exact.

Step 2:

$$\frac{\partial F}{\partial x} = 2xy - x^3$$
$$\frac{\partial F}{\partial y} = x^2$$

Step 3: $F(x, y) = \displaystyle\int 2xy - x^3 \, \partial x = x^2 y - x^4/4 + C(y)$

Step 4: $F(x, y) = \displaystyle\int x^2 \, \partial y = x^2 y + \tilde{C}(x)$

Step 5: $F(x, y) = x^2 y - x^4/4$

Step 6: $x^2 y - x^4/4 = K$

SECOND:

$$1^2 \times 3 - 1^4/4 = K$$
$$11/4 = K$$

So $x^2 y - x^4/4 = 11/4$ or $y = x^2/4 + (11/4)x^{-2}$. (It is always easiest to wait to find an explicit solution until after finding the arbitrary constant when solving an *exact* initial value problem.)

Exercises: Determine which of the following equations are exact (you may first have to write them in the form $M(x,y)+N(x,y)\frac{dy}{dx}=0$). If they are exact, solve them.

(1) $2xy+x^2\frac{dy}{dx}=0$

(2) $e^x+2xy^2+(2x^2y-1/y)\frac{dy}{dx}=0$

(3) $y\cos(xy)+(x\cos(xy)-3y^2)\frac{dy}{dx}=0$

(4) $e^x+y+(e^y+2x)\frac{dy}{dx}=0$

(5) $\frac{dy}{dx}=-\frac{ye^x+y}{e^x+x}$

(6) $\frac{dy}{dx}=\frac{1+2y}{2y-2x}$

(7) $x^2y+2xy^2\frac{dy}{dx}=0$

(8) $4x^3y+x^4\frac{dy}{dx}=0$

(9) $\frac{dy}{dx}=\frac{x+1}{y+1}$

(10) $\frac{dy}{dx}=xy$

(11) $3x^2\tan(y)+1+(x^3\sec^2(y)-1)\frac{dy}{dx}=0,\qquad y(0)=0$

(12) $y+(1+x+2y)\frac{dy}{dx}=0,\qquad y(-1)=2$

(13) $2x+y+2y\frac{dy}{dx}=0,\qquad y(0)=1$

(14) $1/y+(1-x/y^2)\frac{dy}{dx}=\cos(x),\qquad y(\pi)=-1$

(15) $\cos(y)-x\sin(y)\frac{dy}{dx}=0,\qquad y(0)=\pi/2$

(16) $e^{x+y}+(e^{x+y}+e^y)\frac{dy}{dx}=0,\qquad y(0)=0$

(17) $\frac{dy}{dx}=\frac{x+y}{2y-x},\qquad y(0)=1$

(18) $\frac{dy}{dx}=\frac{x+4y}{4x-y},\qquad y(1)=0$

(19) $\frac{dy}{dx}=\frac{e^x-y}{x+\cos(y)},\qquad y(0)=0$

(20) $\frac{dy}{dx}=\frac{\sin(x)}{\sin(y)},\qquad y(0)=\pi/2$

§5 Linear Equations

Discussion: We solved exact equations by assuming there was a solution and working backwards. But there will be a solution to any equation we can solve, and it is a theorem that all (reasonable) first order equations can be solved. So why aren't all equations exact? Consider the following example.

$$(x^2 + y) + (x - \sin(y))\frac{dy}{dx} = 0 \qquad \text{is exact}$$
$$\frac{x^2 + y}{x} + \frac{x - \sin(y)}{x}\frac{dy}{dx} = 0 \qquad \text{is not exact}$$

There is no real difference between the two equations, we have just divided through by x to obtain the second equation from the first. The equations have the same solutions. But when we test for exactness, we find that only the first equation is exact. The reason most first order equations aren't exact is that some key factor has been divided out. One way to solve the equation would be to find the factor, called an integrating factor, and put it back in. Unfortunately, finding integrating factors can be extremely difficult. There is one situation where it is fairly straightforward — when the equation is linear. A first order differential equation is said to be **linear** if it can be written in the form

$$y' + p(x)y = q(x)$$

For a linear equation, we find an integrating factor as follows

$$\mu(x) = e^{\int p(x)\,dx}$$

If we multiply through the linear equation by $\mu(x)$ we obtain

$$\mu(x)y'(x) + \mu(x)p(x)y(x) = \mu(x)q(x)$$

Now $\mu'(x) = \mu(x)p(x)$ from the way we defined $\mu(x)$, so we can now write our equation as

$$\mu(x)y'(x) + \mu'(x)y(x) = \mu(x)q(x)$$

or (applying the product rule in reverse)

$$\frac{d}{dx}(\mu(x)y(x)) = \mu(x)q(x)$$

Now we integrate both sides to obtain

$$\mu(x)y(x) = \int \mu(x)q(x)\,dx$$

so

$$y(x) = \frac{1}{\mu(x)} \int \mu(x)q(x)\,dx$$

WARNING: You can't cancel the $\mu(x)$ inside the integral by the $\mu(x)$ outside the integral. This formula is confusing because we are using x as both the dummy variable of integration and as the independent variable of the resulting function. Because of this it is often preferable to write the solution as

$$y(x) = \frac{1}{\mu(x)} \left(\int_a^x \mu(s)q(s)\,ds + C \right)$$

In this form we use s as the variable of integration and use x as the upper limit of integration to show we want to treat the result as a function of x. The a as the lower limit of integration is an arbitrary constant as is C, the constant of integration. This seems to give us two arbitrary constants, but we really only have one "degree of freedom." That is, for any choice of a, we can choose C to give us any solution to the equation. This is easiest to understand when we deal with a particular example.

Example: $y' + 2y = e^x$

$$\begin{aligned}
\mu(x) &= e^{\int 2\,dx} = e^{2x} \\
y(x) &= \frac{\int_a^x e^{2s}e^s\,ds + C}{e^{2x}} = \frac{\int_a^x e^{3s}\,ds + C}{e^{2x}} \\
y(x) &= \frac{e^{3x}/3 - e^{3a}/3 + C}{e^{2x}} \\
y(x) &= e^x/3 + \tilde{C}e^{-2x} \qquad (\tilde{C} = C - e^{3a}/3)
\end{aligned}$$

As you see, the choice of a doesn't matter, it all gets sucked into the arbitrary constant in the end. Linear equations won't have singular solutions, so the general solution gives all the solutions. You may be wondering about why I left out the constant of integration when computing $\mu(x)$. I could have included a constant of integration, which would give me an infinite family of integrating factors. But I don't need an infinite family of integrating factors to solve the problem; I only need one integrating factor. So I can ignore that constant. We are now ready to give the paradigm. While it is possible to just memorize the formula for the solution, I prefer to work through the whole process. While that takes longer, I find I am less likely to make mistakes if I go through the details. You are welcome to use the paradigm or the formula depending on which you find works best for you.

Paradigm: Solve $dy/dx + 2y/x = 4$

STEP 1: Find the integrating factor.

$$\mu(x) = e^{\int 2/x\, dx} = e^{2\log(x)} = x^2$$

STEP 2: Multiply through by the integrating factor.

$$x^2 \frac{dy}{dx} + 2xy = 4x^2$$

STEP 3: Recognize the left hand side as the derivative of μy.

$$\frac{d}{dx}(x^2 y) = 4x^2$$

STEP 4: Integrate both sides.

$$x^2 y = \int 4x^2\, dx = (4/3)x^3 + C$$

STEP 5: Solve for y.

$$y(x) = (4/3)x + Cx^{-2}$$

EXAMPLE: Solve the initial value problem $dy/dx + 2xy = 1$, $y(1) = 2$.

FIRST: Find the general solution.

Step 1: $\mu(x) = e^{\int 2x\, dx} = e^{x^2}$

Step 2: $e^{x^2} dy/dx + 2xe^{x^2} y = e^{x^2}$

Step 3: $\dfrac{d}{dx}(e^{x^2} y) = e^{x^2}$

Step 4: $e^{x^2} y = \displaystyle\int e^{x^2}\, dx = ???$

Unfortunately, I don't know the indefinite integral of e^{x^2}. So I leave the integral as a definite integral with x as the upper limit to give me a function of x and 1 as the lower limit because the initial value is given at 1 (you'll see why that matters in a second). And I won't forget the constant of integration.

Step 4: (Take Two) $e^{x^2} y = \int_1^x e^{s^2}\, ds + C$

Step 5: $y(x) = e^{-x^2} \int_1^x e^{s^2}\, ds + Ce^{-x^2}$

SECOND: Solve the initial value problem by plugging in.

$$\begin{aligned} y(1) = e^{-1^2} \int_1^1 e^{s^2}\, ds + C &\overset{\text{set}}{=} 2 \\ C &= 2 \end{aligned}$$

While I don't know the indefinite integral of e^{s^2}, I do know that the integral of anything from 1 to 1 is 0. That is the advantage of choosing the lower limit to be the same as the place where the initial value is given.

Exercises: Determine which of the following equations are linear. If they are linear, solve them.

(1) $\dfrac{dy}{dx} + 3y = 2x$

(2) $\dfrac{dy}{dx} + x^3 y = e^y$

(3) $x\dfrac{dy}{dx} + 2y = x\sin(x)$

(4) $\dfrac{dr}{da} + 2ar = e^a$

(5) $\dfrac{dp}{dt} + tp = e^{-t^2/2}$

(6) $e^x\dfrac{dy}{dx} + 2e^x y = 1$

(7) $\dfrac{dx}{dt} + tx = 0, \qquad x(0) = 2$

(8) $\dfrac{dq}{dt} + e^t q = \sin(t), \qquad q(0) = 1$

(9) $2\dfrac{dy}{dx} + 4y^2 = \cos(x)$

(10) $\dfrac{dy}{dx} + 3y/x = x^{-2}, \qquad y(1) = 0$

(11) $\dfrac{dy}{dx} + \dfrac{x}{y} = 0$

(12) $\dfrac{dp}{dt} + 2p = p^2$

(13) $\dfrac{dx}{dt} + 2x = t^2$

(14) $\dfrac{dx}{dt} + 4x = \sin(t)$

(15) $\dfrac{dy}{dx} - y = e^x, \qquad y(0) = 0$

(16) $\dfrac{dy}{dx} + y = x^2, \qquad y(0) = 1$

(17) $\dfrac{dy}{dx} + e^{xy} = x, \qquad y(0) = 2$

(18) $\dfrac{dy}{dx} + \cos(x)y = 1, \qquad y(0) = 1$

(19) $\dfrac{dy}{dx} + 4y = e^{-x}, \qquad y(0) = 0$

(20) $\dfrac{dx}{dt} + x = f(t), \qquad x(0) = 0$

§6 Change of Variables

Discussion: Basically, there is only one way to solve a first order differential equation. That is to convert it to exact form and integrate it. We have applied this to exact equations, which are already in exact form; to separable equations, which are in exact form after they are separated; and to linear equations, which are in exact form after they are multiplied by an integrating factor. There is one other standard approach to putting a first order equation in exact form, making a change of variables. Two instances where this works are Bernoulli equations and homogeneous equations. (WARNING: The term homogeneous has several different meanings in differential equations. We will encounter the term later with a completely different meaning in Chapter 2.)

BERNOULLI EQUATIONS

An equation is a Bernoulli equation if it can be written in the form

$$\frac{dy}{dx} + p(x)y = q(x)y^n$$

for some n. Bernoulli equations are almost linear equations, they just have an extra y^n term. We can make a change of variables to get rid of this term and rewrite the equation as a linear equation. Let $y = v^{1/(1-n)}$. Then $y^n = v^{n/(1-n)}$ and

$$\frac{dy}{dx} = \frac{1}{1-n}v^{1/(1-n)-1}\frac{dv}{dx} = \frac{1}{1-n}v^{n/(1-n)}\frac{dv}{dx}$$

so plugging into our equation we obtain

$$\frac{1}{1-n}v^{n/(1-n)}\frac{dv}{dx} + p(x)v^{1/(1-n)} = q(x)v^{n/(1-n)}$$

Dividing through by $[1/(1-n)]v^{n/(1-n)}$ we obtain

$$\frac{dv}{dx} + (1-n)p(x)v = (1-n)q(x)$$

which is linear. We now solve the linear equation using ordinary techniques. Finally, since the original problem is stated in terms of x and y, the answer should be given in terms of x and y — not v. So we undo our substitution by writing $y = v^{1/(1-n)}$. We must also check for singular solutions of the form $y = 0$ since the substitution of $y = v^{1/(1-n)}$ is equivalent to dividing by y if $n > 1$.

Paradigm: $\dfrac{dy}{dx} + 3y = e^x y^2$.

STEP 1: Make the substitution $y = v^{1/(1-n)}$.

Here $n = 2$ so $y = v^{-1}$

$$\frac{dy}{dx} = -v^{-2}\frac{dv}{dx}$$
$$y^2 = v^{-2}$$

and the equation is

$$-v^{-2}\frac{dv}{dx} + 3v^{-1} = e^x v^{-2}.$$

STEP 2: Divide through to obtain a linear equation

$$\frac{dv}{dx} - 3v = -e^x.$$

STEP 3: Solve the linear equation for v

SubStep 1: $\mu(x) = e^{\int -3\,dx} = e^{-3x}$

SubStep 2: $e^{-3x}\frac{dv}{dx} - 3e^{-3x}v = -e^{-2x}$

SubStep 3: $\frac{d}{dx}(e^{-3x}v) = -e^{-2x}$

SubStep 4: $e^{-3x}v = \int -e^{-2x}\,dx = (1/2)e^{-2x} + C$

SubStep 5: $v = (1/2)e^x + Ce^{3x}$

STEP 4: Back substitute to find y

$$y = v^{-1} = [(1/2)e^x + Ce^{3x}]^{-1}$$

STEP 5: Check for singular solution $y = 0$.

$$0 + 3 \cdot 0 = e^x \cdot 0$$

so $y = 0$ is a solution as well.

EXAMPLE: $\dfrac{dy}{dx} + y = \cos(x)/y, \qquad y(0) = 1$

FIRST: Find the general solution.

Step 1: Let $y = v^{1/2}$ then

$$(1/2)v^{-1/2}\frac{dv}{dx} + v^{1/2} = \cos(x)/v^{1/2}$$

Step 2: $\dfrac{dv}{dx} + 2v = 2\cos(x)$

Step 3: Solving for v

SubStep 1: $\mu(x) = e^{\int 2\,dx} = e^{2x}$

SubStep 2: $e^{2x}\dfrac{dv}{dx} + 2e^{2x}v = 2e^{2x}\cos(x)$

SubStep 3: $\dfrac{d}{dx}(e^{2x}v) = 2e^{2x}\cos(x)$

SubStep 4: $e^{2x}v = \int 2e^{2x}\cos(x)\,dx = e^{2x}\left(\frac{4}{5}\cos(x) + \frac{2}{5}\sin(x)\right) + C$

SubStep 5: $v(x) = \left(\dfrac{4}{5}\cos(x) + \dfrac{2}{5}\sin(x)\right) + Ce^{-2x}$

Step 4: $y(x) = \sqrt{\dfrac{4}{5}\cos(x) + \dfrac{2}{5}\sin(x) + Ce^{-2x}}$

Step 5: $y = 0$ is not a solution (can't have $\cos(x)/0$ on right hand side).

SECOND: Plug in the initial value and solve for the arbitrary constant.

$$y(0) = \sqrt{\frac{4}{5}\cos(0) + \frac{2}{5}\sin(0) + Ce^{0}} \stackrel{\text{set}}{=} 1$$

$$\sqrt{\frac{4}{5} + C} = 1$$

$$C = \frac{1}{5}$$

So $y(x) = \sqrt{\dfrac{4}{5}\cos(x) + \dfrac{2}{5}\sin(x) + \dfrac{1}{5}e^{-2x}}$

HOMOGENEOUS EQUATIONS

An equation is homogeneous if it can be written in the form

$$\frac{dy}{dx} = f(y/x)$$

for some function f. Usually, it takes some algebraic manipulation to convert the equation to this form. Often, the equation is given in the form

$$\frac{dy}{dx} = \frac{a_n x^n + a_{n-1}x^{n-1}y + a_{n-2}x^{n-2}y^2 + \ldots + a_0 y^n}{b_n x^n + b_{n-1}x^{n-1}y + b_{n-2}x^{n-2}y^2 + \ldots + b_0 y^n}$$

In this case, divide through by x^n to obtain the desired form. Once the equation is in the desired form, we make the change of variables $v = y/x$ so that $y = xv$ and $dy/dx = xdv/dx + v$. Plugging into our equation we obtain

$$x\frac{dv}{dx} + v = f(v)$$

which is separable. We now solve the separable equation using ordinary techniques. Finally, since the original problem is stated in terms of x and y, the answer should be given in terms of x and y — not v. So we undo our substitution by writing $y = xv$.

Paradigm: $\dfrac{dy}{dx} = \dfrac{2xy}{x^2 + y^2}$

STEP 0: Convert to homogeneous form

In this case, every term has order 2 so we divide by x^2 to obtain

$$\frac{dy}{dx} = \frac{2(y/x)}{1 + (y/x)^2}$$

STEP 1: Make the substitution $v = y/x$

This is equivalent to $y = xv$ so we get $dy/dx = x\,dv/dx + v$ and plugging into the equation yields

$$x\frac{dv}{dx} + v = \frac{2v}{1 + v^2}$$

STEP 2: Solve the separable equation

SubStep 1: Separate the variables

$$\begin{aligned} x\frac{dv}{dx} &= \frac{2v}{1+v^2} - v = \frac{v - v^3}{1 + v^2} \\ \frac{(1+v^2)dv}{v - v^3} &= \frac{dx}{x} \end{aligned}$$

SubStep 2: Integrate both sides

$$\log(\frac{v}{v^2 - 1}) = \log(x) + C$$

SubStep 3: Solve for v

$$\begin{aligned} \frac{v}{v^2 - 1} &= kx \\ v &= \frac{1 \pm \sqrt{1 + 4k^2x^2}}{2kx} \end{aligned}$$

SubStep 4: Check for singular solutions.

We divided by $v^2 - 1$ which is 0 at $v = \pm 1$. Neither of these is included in the general solution so they are both singular solutions.

STEP 3: Back Substitute $v = y/x$ to find y

$$y = \frac{1 \pm \sqrt{1 + 4k^2x^2}}{2k}$$

is the general solution and the singular solutions are

$$y = \pm x.$$

EXAMPLE: $\dfrac{dy}{dx} = \dfrac{x+y}{y}, \qquad y(1) = 1/2 + \sqrt{5}/2$

FIRST: Find the general solution.

Step 1: Let $v = y/x$ (or $y = xv$) to get

$$x\frac{dv}{dx} + v = \frac{v+1}{v}$$

Step 2: We now solve this equation.

SubStep 1:

$$\begin{aligned} x\frac{dv}{dx} &= \frac{v+1}{v} - v \\ x\frac{dv}{dx} &= \frac{v+1-v^2}{v} \\ \frac{v\,dv}{v+1-v^2} &= \frac{dx}{x} \end{aligned}$$

SubStep 2:

$$\int \frac{v\,dv}{v+1-v^2} = \int \frac{dx}{x}$$

$$\left(\frac{\sqrt{5}}{10} - \frac{1}{2}\right)\log(2v + \sqrt{5} - 1) - \left(\frac{\sqrt{5}}{10} + \frac{1}{2}\right)\log(2v - \sqrt{5} - 1) = \log(x) + C$$

$$\frac{(2v + \sqrt{5} - 1)^{\sqrt{5}/10 - 1/2}}{(2v - \sqrt{5} - 1)^{\sqrt{5}/10 + 1/2}} = kx$$

SubStep 3: I can't solve this for v to get an explicit solution.

SubStep 4: We divided by $v + 1 - v^2$ whose roots are $v = 1/2 \pm \sqrt{5}/2$. These are both solutions to the equation and only $v = 1/2 - \sqrt{5}/2$ is an example of the general solution (corresponding to $k = 0$), so $v = 1/2 + \sqrt{5}/2$ is a singular solution.

Step 3: Replace v by y/x to get

$$\frac{(2y/x + \sqrt{5} - 1)^{\sqrt{5}/10 - 1/2}}{(2y/x - \sqrt{5} - 1)^{\sqrt{5}/10 + 1/2}} = kx$$

with a singular solution $y/x = 1/2 + \sqrt{5}/2$.

SECOND: Plug in the initial value and solve for the arbitrary constant.

$$\begin{aligned} \frac{(2 \times (1/2 + \sqrt{5}/2)/1 + \sqrt{5} - 1)^{\sqrt{5}/10 - 1/2}}{(2 \times (1/2 + \sqrt{5}/2)/1 - \sqrt{5} - 1)^{\sqrt{5}/10 + 1/2}} &= k \times 1 \\ \frac{(2\sqrt{5})^{\sqrt{5}/10 - 1/2}}{0} &= k \end{aligned}$$

Well that didn't work out very well. There is no way to choose k to get an example of the general solution which satisfies the initial condition. But we have one more solution left to consider. The singular solution is $y/x = 1/2 + \sqrt{5}/2$ and it does satisfy $y(1) = 1/2 + \sqrt{5}/2$. So the solution to our initial value problem is

$$y = \frac{(1 + \sqrt{5})x}{2}$$

This problem involves somewhat more complicated numbers than most in this text. That has nothing to do with the solution to the initial value problem being the singular solution. Homogeneous equations often give rise to integrals with ugly answers.

Exercises: Determine which of the following equations are either Bernoulli or homogeneous. If they are Bernoulli or homogeneous, solve them.

(1) $\frac{dy}{dx} + 2y = 3xy^2$

(2) $\frac{dy}{dx} = \frac{x+y}{2xy}$

(3) $\frac{dy}{dx} = \frac{x+y}{2x+3y} 2x + y$

(4) $\frac{ds}{dr} - s = e^r \sqrt{s}$

(5) $\frac{dp}{dt} + 3tp^2 = t^2 p^3$

(6) $\frac{dy}{dx} + y/x = x/y^2$

(7) $\frac{dy}{dx} = 3y - x/y, \qquad y(1) = 1$

(8) $\frac{dv}{ds} = \frac{6v - s}{v + 4s}, \qquad v(0) = 1$

(9) $\frac{dy}{dx} = \frac{2y + 3x}{2x - y}, \qquad y(0) = 1$

(10) $\frac{dy}{dx} + y = \sin(x) y^3$

(11) $\frac{dy}{dx} + 4y = y^2$

(12) $\frac{dy}{dx} + 4x^3 y = xy^3$

(13) $\frac{dy}{dx} = \frac{y^3}{2x^2 y + x^3}$

(14) $\frac{dy}{dx} = \frac{x^2 + y^2}{2x^2}$

(15) $\frac{dy}{dx} = \frac{y}{x - 4y}, \qquad y(0) = 1$

(16) $\frac{dy}{dx} - y = x + y^2, \qquad y(1) = 2$

(17) $\frac{dy}{dx} + y = \frac{x^2 + y^2}{x^2 - y^2}, \qquad y(0) = 0$

(18) $\frac{dy}{dx} + 2y = e^y, \qquad y(0) = 0$

(19) $\frac{dy}{dx} + y = \frac{x^2}{y^2}, \qquad y(0) = 2$

(20) $\frac{dy}{dx} = \frac{\cos(x) + \sin(y)}{\sin(x) + \cos(y)}, \quad y(0) = \pi$