Written Assignment #1: Separable Equations (Solutions)

1. Find $y(1)$ if $\dfrac{dy}{dx} = \dfrac{4e^x}{2y+2}$ and $y(0) = -3$.

   Solution

   First: Find all solutions.

$$\int (2y+2)\,dy = \int 4e^x\,dx \Rightarrow y^2 + 2y = 4e^x + C$$
$$\Rightarrow (y+1)^2 - 1 = 4e^x + C \Rightarrow y = -1 \pm \sqrt{4e^x + C}$$

   Second: Use the initial condition to determine the value for C and which root to use.

$$y(x) = -1 \pm \sqrt{4e^x + C} \Rightarrow y(0) = -1 \pm \sqrt{4e^0 + C} = -3$$
$$\Rightarrow 4 + C = 4 \Rightarrow C = 0.$$

   We now know that $y(x) = -1 \pm \sqrt{4e^x} = -1 \pm 2e^{\frac{1}{2}x}$, but we still need to decide which root to use. Which one satisfies the initial condition?

$$y(x) = -1 + 2e^{\frac{1}{2}x} \Rightarrow y(0) = -1 + 2 = 1 \neq -3 \text{ (no good).}$$

$$y(x) = -1 - 2e^{\frac{1}{2}x} \Rightarrow y(0) = -1 - 2 = -3 \text{ (this is good).}$$

   The solution we want is $y(x) = -1 - 2e^{\frac{1}{2}x}$. Now, we can compute

$$\boxed{y(1) = -1 - 2e^{\frac{1}{2}}}.$$

2. (a) Newton's Law of Cooling (or Heating) provides a simple model for the change in an object's temperature in a surrounding environment with a constant temperature. In words, the law states that **the instantaneous rate of change of the temperature of an object is proportional to the difference in temperature between the object and its surrounding environment**. Write a differential equation that expresses Newton's Law of Cooling mathematically. Use $k > 0$ for the constant of proportionality, β for the *constant* temperature of the surrounding environment, T for the temperature of the object, and t for time.

Solution

Mathematically, the instantaneous rate of change in T (temperature) with respect to t (time) is $\frac{dT}{dt}$. Newton's Law of Cooling states that this should be proportional to $(T - \beta)$ (difference in the object's and environment's temperature). So the differential equations is

$$\boxed{\frac{dT}{dt} = -k(T - \beta)}.$$

How did we decide to use a $-$ sign in front of k? We ask ourselves what happens if the object's temperature is higher than the temperature of the surrounding environment $(T > \beta)$. The object should cool off; i.e. its temperature should decrease; i.e. $\frac{dT}{dt}$ should be negative. Since $k > 0$ and $T - \beta > 0$, in this situation $k(T - \beta)$ is positive, and we must use the negative of it in our differential equation.

(b) Suppose that the constant of proportionality in Newton's Law is .08/hr for coffee. In a room with a constant temperature of 60° F, the temperature of a cup of coffee is measured to be 200° F. What is temperature of the coffee 3 hours later?

Solution

We first decide for what value of t is the coffee 200° F. We may as well say that this occurs when $t = 0$. Plugging in numbers, here is the mathematical problem to solve

$$\frac{dT}{dt} = -.08(T - 60); \quad T(0) = 200.$$

Now, we solve this initial value problem.

$$\int \frac{1}{T - 60}\, dT = \int (-.08) dt \Rightarrow \ln|T - 60| = -.08t + C$$
$$\Rightarrow T = Ce^{-.08t} + 60.$$

Using the initial condition, we find that

$$T(0) = C + 60 = 200 \Rightarrow C = 140.$$

Thus

$$T(t) = 140e^{-.08t} + 60.$$

Since we used $k = .08/\text{hr}$, time is measured in hours. The temperature of the coffee after 3 hours is the value of $T(3)$

$$T(3) = 140e^{-.24} + 60 \approx \boxed{170.1^\circ \text{ F}}.$$

(c) A forensic specialist measured the temperature of a murder victim's body body at 11:00pm and found it to be 90.1° F. At 11:30pm, the temperature was 89.5° F. Assuming the victim's normal body temperature was 98.6° F, at what time was the murder committed if the temperature of the room was a constant 70° F?

<u>Solution</u>

For this problem, we are not given the constant of proportionality; we have to determine it from the information given. First, we need to decide when $t = 0$ occurs and our units for time. Let say that $t = 0$ at 11:00pm, and we will measure time in hours. Mathematically, the information that we have is

$$\frac{dT}{dt} = -k(T - 70); \quad T(0) = 90.1 \text{ and } T(.5) = 89.5.$$

We want to know when did $T = 98.6$, this will be the time of death. First we solve the differential equation (remember that k is a constant; it does not change with t).

$$\int \frac{1}{T - 70}\, dT = \int (-k)\, dt \Rightarrow \ln|T - 70| = -kt + C$$
$$\Rightarrow T = Ce^{-kt} + 70.$$

There are two unknowns in this equation, so we need to find two equations

$$\begin{aligned} &\text{Equation 1:} \quad T(0) = 90.1 \Rightarrow C + 70 = 90.1 \\ &\text{Equation 2:} \quad T(.5) = 89.5 \Rightarrow Ce^{-.5k} + 70 = 89.5. \end{aligned}$$

From the Equation 1, we see that $C = 20.1$, plugging this into Equation 2 yields

$$20.1e^{-.5k} + 70 = 89.5 \Rightarrow e^{-.5k} = \frac{19.5}{20.1} \Rightarrow -.5k = \ln\left(\frac{19.5}{20.1}\right)$$
$$\Rightarrow k = -2\ln\left(\frac{19.5}{20.1}\right).$$

We now have C and k, putting them into the formula for T gives us

$$T(t) = (20.1)e^{2\ln\left(\frac{19.5}{20.1}\right)t} + 70.$$

For what value of t is T equal to 98.6?

$$(20.1)e^{2\ln\left(\frac{19.5}{20.1}\right)t} + 70 = 98.6 \Rightarrow e^{2\ln\left(\frac{19.5}{20.1}\right)t} = \frac{28.6}{20.1}$$

$$\Rightarrow t = \frac{\ln\left(\frac{28.6}{20.1}\right)}{2\ln\left(\frac{19.5}{20.1}\right)} \approx -5.819 \text{ hours.}$$

The victim's body had a temperature of 98.6° F 5 hours and 49 minutes before 11:00pm, so the murder was committed at about $\boxed{\text{5:11pm}}$.