Math240 | Fall 2003
Elementary Differential Equations | Kansas State University

Extra Credit Assignment #2: Boundary Value Problems

Due in recitation Thursday, October 30, 2003

You are encouraged to collaborate with your colleagues. For credit, however, your final write-up must be done individually. Show all your work and make your presentation comprehensible.

In this assignment, we will look at problems where the values of the solution (and possibly its derivatives) at the endpoints are specified. These types of problems are called boundary or endpoint value problems. For boundary value problems, there may be only one trivial solution or infinitely many nontrivial solutions.

1 Eigenvalue Problems

Consider a problem of the form

$$y'' + a_1(x)y' + \lambda a_0(x)y = 0; \quad y(a) = 0, \ y(b) = 0.$$

Thus the values of the solution are specified at the endpoints a and b. Given a_0 and a_1, the question is **for what values of λ is there a nontrivial (nonzero) solution to this problem?** Such values of λ are called *eigenvalues* for the problem, and the associated nontrivial solution is called an *eigenfunction* for the problem. An eigenvalue problem is to find the eigenvalues (and possibly the eigenfunctions) for a differential equation with boundary values specified.

1.1 Example

Determine all of the eigenvalues and associated eigenfunctions for the boundary value problem

$$y'' + \lambda y = 0; \quad y(0) = 0, \ y(1) = 0.$$

We want to find all λ's such that there are nontrivial solutions to this problem. We have three cases

Case 1 ($\lambda = 0$). For this case, the problem reduces to

$$y'' = 0; \quad y(0) = 0, \ y(1) = 0.$$

and the only solution can be found to be zero (the trivial solution) for this problem.

Case 2 ($\lambda < 0 = -\alpha^2$). In this case, the problem becomes

$$y'' - \alpha^2 y = 0; \quad y(0) = 0, \ y(1) = 0.$$

and again the only solution found is the zero (trivial) solution.
Case 3 ($0 < \alpha^2 = \lambda$) In this last case, the problem becomes

$$y'' + \alpha^2 y = 0; \quad y(0) = 0, \ y(1) = 0.$$

The general solution for this problem is $y = C_1 \cos(\alpha x) + C_2 \sin(\alpha x)$. The condition $y(0) = 0$ implies that $C_1 = 0$, thus $y = C_2 \sin(\alpha x)$. The condition that $y(1) = 0$ implies that

$$C_2 \sin(\alpha) = 0.$$

This implies that either $C_2 = 0$ which gives the trivial (zero) solution, or $\sin(\alpha) = 0$. It follows that we have nontrivial solutions when $\alpha = k\pi$, with $k = \pm 1, \pm 2, ...$ Thus the eigenvalues are $\lambda = \alpha^2 = k2\pi^2$, with $k = \pm 1, \pm 2, ...$, and the eigenfunctions are $\sin(k\pi x)$. Noting that $\sin(-k\pi x) = -\sin(k\pi x)$, we can restrict ourselves to positive values for k. In other words, the only nontrivial (nonzero) solutions to

$$y'' + \lambda y = 0; \quad y(0) = 0, \ y(1) = 0.$$

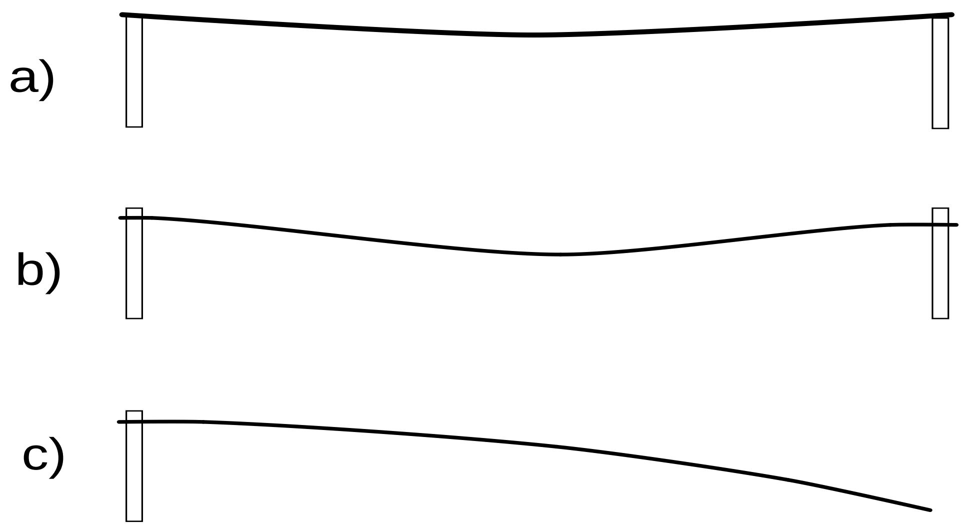
exist when $\lambda = k^2\pi^2$, with $k = \pm 1, \pm 2, ...$, and the associated eigenfunction is $\sin(k\pi x)$. We do not need the negative values of k since $\sin(-k\pi x) = (-1) \cdot \sin(k\pi x)$

2 Deflection of a Uniform Beam

A beam of homogeneous material that has a uniform cross section and is supported at one (or both) of its ends is distorted by its own weight along its horizontal axis. The center of the shape of the beam can be represented by the graph of a function $y = y(x)$, which is called the *deflection curve*. If this deflection curve is relatively small in magnitude, then as a consequence of the theory of elasticity, it satisfies the fourth-order differential equation

$$EIy^{(4)} = w.$$

Here E is the Young's modulus, I is a measure of the inertia of the cross-section of the beam, and w is the beam's weight per unit of length. These are constants that depend only on the material of which the beam is made. To determine a particular solution to this equation, we need to specify 4 conditions on the solution. The following diagram illustrates the possible boundary values that can arise for this type of problem.



a) simply supported beam $y(0) = y''(0) = y(L) = y''(L) = 0$;
b) built-in ends $y(0) = y'(0) = y(L) = y'(L) = 0$;
c) right-end free $y(0) = y'(0) = 0$ and $y''(L) = y^3(L) = 0$.

Here $x = 0$ at the left-end of the beam, and $x = L$ at the right-end of the beam. In a), the beam is just sitting atop two supports, and the boundary conditions can be reduced to $y(0) = y''(0) = 0$ and $y(L) = y''(L) = 0$. In b), the beam is fixed into walls (immobile) at the ends, and the boundary condition can be reduced to $y(0) = y'(0) = 0$ and $y(L) = y''(L) = 0$. In c), only one end of the beam is built into a wall while the other is free, and the boundary conditions can be reduced to $y(0) = y'(0) = 0$ and $y''(L) = y^3(L) = 0$ (these are the boundary conditions for a *cantilever*).

3 Problems

Solve the following problems.

1. Find the eigenvalues and eigenfunctions for

$$y'' + \lambda y = 0; \quad y(-\pi) = 0 \text{ and } y(\pi) = 0.$$

2. Consider the eigenvalue problem

$$y'' + 2y' + \lambda y = 0; \quad y(0) = y(1) = 0.$$

(a) Show that $\lambda = 1$ is not an eigenvalue.

(b) Show that there is no eigenvalue λ such that $\lambda < 1$.

(c) Show that $\lambda_k = k^2\pi 2 + 1$ is an eigenvalue. What is its associated eigenfunction?

3. Suppose that the length of a beam is 600 cm., its Young's modulus is $E = 2 \times 10^{12}$ g/(cm·sec^2), its weight per unit length is 38000 dynes/cm, and its moment of inertia (of a cross-section about the curve through the centroid) is $I = 2\text{cm}^4$. What is the the deflection curve for this beam if it is sitting atop two supports.

4. Suppose that the length of a *cantilever* is 600 cm., its Young's modulus is $E = 2 \times 10^{12}$ g/(cm·sec^2), its weight per unit length is 38000 dynes/cm, and its moment of inertia (of a cross-section about the curve through the centroid) is $I = 2\text{cm}^4$. What is the maximum deflection of the beam?