Math 221 - Analytic Geometry and Calculus 2
Midterm Exam, Summer 2009

Name ____________ Instructor Naeem

Show all work and box your final answers. Good luck !

1. (a)(10 points each) Evaluate $\int xe^{-x}dx$

$$= -xe^{-x} - \int -e^{-x} \cdot 1\, dx$$

$$= \boxed{-xe^{-x} - e^{-x} + C}$$

(b) Evaluate $\int p^5 \ln p \; dp$

$$= \ln p \cdot \frac{p^6}{6} - \int \frac{p^{\cancel{6}\,5}}{6} \frac{1}{\cancel{p}} dp$$

$$= \boxed{\frac{p^6 \ln p}{6} - \frac{p^6}{36} + C}$$

2.(a) (10 points each) Evaluate $\int \sin^6 x \cos^3 x \, dx$

$$= \int \sin^6 x \,(1 - \sin^2 x) \cos x \, dx$$

$u = \sin x$

$du = \cos x \, dx$

$$= \int u^6 (1 - u^2) \, du$$

$$= \int (u^6 - u^8) \, du = \frac{u^7}{7} - \frac{u^9}{9} + C$$

$$= \boxed{\frac{\sin^7 x}{7} - \frac{\sin^9 x}{9} + C}$$

(b) Evaluate $\int_0^{\pi/4} \sec^4 \theta \tan^4 \theta \, d\theta$

$$= \int_0^{\pi/4} \tan^4 \theta \,(1 + \tan^2 \theta) \sec^2 \theta \, d\theta$$

$u = \tan\theta \Rightarrow du = \sec^2 \theta \, d\theta$

$\theta = 0 \Rightarrow u = 0$; $\theta = \pi/4 \Rightarrow u = 1$

$$= \int_0^1 u^4 (1 + u^2) \, du = \int_0^1 (u^4 + u^6) \, du$$

$$= \left(\frac{u^5}{5} + \frac{u^7}{7}\right)\Bigg|_0^1 = \frac{1}{5} + \frac{1}{7} = \boxed{\frac{12}{35}}$$

3. (a) (10 points each) Evaluate $\lim_{t\to 0} \frac{e^{3t}-1}{t}$

$\left(\frac{0}{0}\right)$

$$\overset{H}{=} \lim_{t\to 0} \frac{3e^{3t}}{1} = 3e^0 = \boxed{3}$$

(b) Evaluate $\lim_{x\to\infty} x^{1/x}$

$$\ln\left(\lim_{x\to\infty} x^{1/x}\right) = \lim_{x\to\infty} \frac{\ln x}{x} \quad \left(\frac{\infty}{\infty}\right)$$

$$= \lim_{x\to\infty} \frac{1/x}{1} = 0$$

$$\Rightarrow \lim_{x\to\infty} x^{1/x} = e^0 = \boxed{1}$$

4. (a) (10 points) Evaluate $\int_e^{\infty} \frac{1}{x(\ln x)^3} dx$ $= \lim_{t \to \infty} \int_e^t \frac{1}{x(\ln x)^3} dx$

$u = \ln x \Rightarrow du = \frac{1}{x} dx$

$x = e \Rightarrow u = 1$, $x = t \Rightarrow u = \ln t$

$$\int_e^{\infty} \frac{1}{x(\ln x)^3} dx = \lim_{t \to \infty} \int_1^{\ln t} \frac{du}{u^3}$$

$$= \lim_{t \to \infty} \left(\frac{1}{-2u^2} \Bigg|_1^{\ln t} \right)$$

$$= \lim_{t \to \infty} \left[\frac{1}{-2(\ln t)^2} - \frac{1}{-2(1)^2} \right] = \boxed{\frac{1}{2}}$$

(b) (5 points) Set up, but do not evaluate, an integral for the area of the surface obtained by rotating the curve

$$y = x^4, \ 0 \le x \le 1$$

about the x-axis.

$$2\pi \int_0^1 y \sqrt{1 + \left(\frac{dy}{dx}\right)^2} \, dx$$

$$= 2\pi \int_0^1 x^4 \sqrt{1 + (4x^3)^2} \, dx$$

$$\boxed{2\pi \int_0^1 x^4 \sqrt{1 + 16x^6} \, dx}$$

5. (a) (10 points) Evaluate $\displaystyle\int \frac{x-9}{(x+5)(x-2)}dx$

$$= \int \left(\frac{A}{x+5} + \frac{B}{x-2} \right) dx$$

$$= \int \left(\frac{2}{x+5} - \frac{1}{x-2} \right) dx$$

$$= \boxed{2 \ln|x+5| - \ln|x-2| + C}$$

(b) (5 points) Set up, but do not evaluate, an integral for the length of the curve

$$y = xe^{-x^2}, \quad 0 \le x \le 1.$$

$$\int_0^1 \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$$

$$= \boxed{\int_0^1 \sqrt{1 + \left(e^{-x^2} - 2x^2e^{-x^2}\right)^2}\, dx}$$

6. (a) (10 points each) Find the length of the curve

$$y = 1 + 6x^{3/2},\ 0 \le x \le 1.$$

$$\int_0^1 \sqrt{1 + \left(\cancel{8}\cdot\frac{3}{\cancel{4}}\, x^{1/2}\right)^2}\, dx$$

$$= \int_0^1 \sqrt{1 + 81x}\, dx$$

$$= \left.\frac{(1+81x)^{3/2}}{81 \cdot 3/2}\right|_0^1$$

$$= \boxed{\frac{2}{243}\left[(82)^{3/2} - 1\right]}$$

(b) Find the area of the surface obtained by rotating the curve

$$y = x^3,\ 0 \le x \le 2$$

about the x-axis.

$$2\pi \int_0^2 x^3 \sqrt{1 + (3x^2)^2}\, dx$$

$$2\pi \frac{1}{36} \int_0^2 36x^3 \sqrt{1 + 9x^4}\, dx$$

$$2\pi \frac{1}{36} \cdot \left.\frac{2}{3}(1 + 9x^4)^{3/2}\right|_0^2$$

$$= \boxed{\frac{\pi}{27}\left[(145)^{3/2} - 1\right]}$$

7. (10 points each) Eliminate the parameter to find a Cartesian equation of the curve

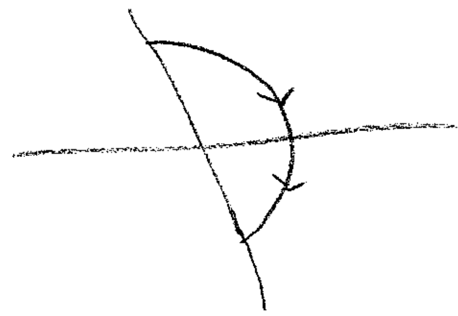
$$x = \sin\theta,\ y = \cos\theta, \quad 0 \le \theta \le \pi.$$

Sketch the curve and indicate with an arrow the direction in which the curve is traced as the parameter increases.

$x^2 + y^2 = \sin^2\theta + \cos^2\theta$

$x^2 + y^2 = 1$

$x \geq 0$



(b) Find an equation of the tangent to the curve

$$x = t - t^{-1},\ y = 1 + t^2$$

at the point corresponding to $t = 1$.

$x(1) = 0 \quad , \quad y(1) = 2$

$\frac{dy}{dx} = \frac{2t}{1 + t^{-2}} \Rightarrow \frac{dy}{dx}(1) = 1$

$y - 2 = 1(x - 0)$

$\Rightarrow \boxed{y = x + 2}$

8. (10 points each) For which values of t is the curve

$$x = t^3 - 12t, \ y = t^2 - 1$$

concave upward?

$$\frac{dy}{dx} = \frac{2t}{3t^2 - 12}$$

$$\frac{d^2y}{dx^2} = \frac{(2 \cdot (3t^2-12) - 2t \cdot (6t)) / (3t^2-12)^2}{3t^2 - 12}$$

$$= \frac{\cancel{6t^2} - 24 - \cancel{6t^2}}{(3t^2 - 12)^3}$$

$$= \frac{-24}{27(t-2)(t+2)}$$

concave up if $-2 < t < 2$.

(b) Find the exact area of the surface obtained by rotating the curve

$$x = 3t - t^3, \ y = 3t^2, \quad 0 \le t \le 1$$

about the x-axis.

$$2\pi \int_0^1 3t^2 \sqrt{(3 - 3t^2)^2 + (6t)^2} \, dt$$

$$= 6\pi \int_0^1 t^2 \sqrt{(3 + 3t^2)^2} \, dt$$

$$= 6\pi \int_0^1 (3t^2 + 9t^4) \, dt = 3\left(t^3 + \frac{9}{5}t^5\right)\Big|_0^1$$

$$= 6\pi\left(1 + \frac{9}{5}\right) = \boxed{\frac{84\pi}{5}}$$