Math 221 - Analytic Geometry and Calculus 2
Final Exam, Summer 2009

Name ______ Instructor Naeem

Show all work and box your final answers. Books, notes or calculators are not allowed. Good luck !

1. (10 points each) (a) Evaluate $\lim_{x\to 0} \frac{\sin 7x - 7x}{\cos 3x - 1}$ $\left(\frac{0}{0}\right)$

$$\overset{H}{=} \lim_{x\to 0} \frac{7\cos 7x - 7}{-3\sin 3x} \left(\frac{0}{0}\right)$$

$$\overset{H}{=} \lim_{x\to 0} \frac{-49\sin 7x}{-9\cos 3x}$$

$$= \frac{-49(0)}{-9(1)} = \boxed{0}$$

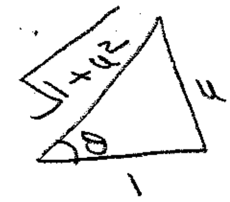
(b) Evaluate $\lim_{t\to 0} \frac{t^2}{e^{3t} - 3t - 1}$ $\left(\frac{0}{0}\right)$

$$\overset{H}{=} \lim_{t\to 0} \frac{2t}{3e^{3t} - 3} \left(\frac{0}{0}\right)$$

$$\overset{H}{=} \lim_{t\to 0} \frac{2}{9e^{3t}} = \boxed{\frac{2}{9}}$$

2. (10 points each) (a) Evaluate $\int \frac{\sqrt{1+u^2}}{u}\,du$

$u = \tan\theta$
$du = \sec^2\theta\, d\theta$



$$= \int \frac{\sec\theta}{\tan\theta} \sec^2\theta\, d\theta$$

$$= \int \frac{\sec\theta}{\tan\theta} (1+\tan^2\theta)\, d\theta$$

$$= \int \left(\frac{1}{\cancel{\cos\theta}} \cdot \frac{\cancel{\cos\theta}}{\sin\theta} + \frac{\sec\theta \tan^{\cancel{2}}\theta}{\cancel{\tan\theta}} \right) d\theta$$

$$= \int (\csc\theta + \sec\theta\tan\theta)\, d\theta$$

$$= \ln|\csc\theta - \cot\theta| + \sec\theta + C$$

$$= \boxed{\ln\left|\frac{\sqrt{1+u^2}}{u} - \frac{1}{u}\right| + \sqrt{1+u^2} + C}$$

(b) Evaluate $\int \frac{x^3-2x-4}{x^2-4}\,dx$

$$\frac{x^3-2x-4}{x^2-4} = x + \frac{2x-4}{x^2-4} \quad \text{(long division)}$$

$$= x + \frac{2\cancel{(x-2)}}{\cancel{(x-2)}(x+2)}$$

$$= x + \frac{2}{x+2}$$

$$\int \frac{x^3-2x-4}{x^2-4}\,dx = \boxed{\ln|x| + 2\ln|x+2| + C}$$

3. (10 points each) (a) Evaluate $\int \cos^2 x \, e^{\cos x} \sin x dx$

$u = \cos x$
$du = \sin x dx$

$$= -\int u^2 e^u du = -\left[u^2 e^u - 2\int u e^u du\right]$$

$$= -u^2 e^u + 2\left[u e^u - \int e^u du\right]$$

$$= -u^2 e^u + 2u e^u - 2e^u + C$$

$$= \boxed{-\cos^2 x \, e^{\cos x} + 2\cos x \, e^{\cos x} - 2e^{\cos x} + C}$$

(b) Evaluate $\int e^x \sin^5(e^x) \cos^2(e^x) dx$ $\quad u = e^x \Rightarrow du = e^x dx$

$$= \int \sin^5(u) \cos^2(u) du$$

$$= \int (1 - \cos^2(u))^2 \cos^2(u) \sin(u) du$$

$w = \cos(u)$
$dw = -\sin(u) du$

$$= \int -(1 - w^2)^2 w^2 dw$$

$$= \int -(w^2 + w^6 - 2w^4) dw = -\frac{w^3}{3} - \frac{w^7}{7} + \frac{2}{5} w^5 + C$$

$$= \boxed{-\frac{\cos^3(e^x)}{3} - \frac{\cos^7(e^x)}{7} + \frac{2}{5}\cos^5(e^x) + C}$$

4. (10 points each) (a) Find the exact area of the surface obtained by rotating the curve $x = 3t^2$, $y = 3t - t^3$, $0 \le t \le 1$ about the x-axis.

$$A = 2\pi \int_0^1 y \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

$$= 2\pi \int_0^1 (3t - t^3) \sqrt{(6t)^2 + (3 - 3t^2)^2}\, dt$$

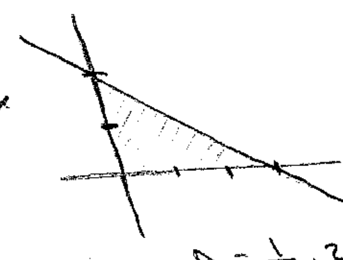
$$= 2\pi \int_0^1 (3t - t^3) \sqrt{(3 + 3t^2)^2}\, dt$$

$$= 2\pi \int_0^1 (9t + 6t^3 - 3t^5)\, dt = 6\pi \int_0^1 (3t + 2t^3 - t^5)\, dt$$

$$= 6\pi \left(\frac{3}{2}t^2 + \frac{1}{2}t^4 - \frac{1}{6}t^6\right)\Big|_0^1 = 3\pi \left(3 + 1 - \frac{1}{3}\right)$$

$$= \boxed{11\pi}$$

(b) Find the exact coordinates of the centroid of the region bounded by the curves $2x + 3y = 6$, $y = 0$, $x = 0$.



$$A = \frac{1}{2} \cdot 2 \cdot 3 = 3$$

$$M_x = \frac{1}{2} \int_0^3 y^2\, dx = \frac{1}{2} \int_0^3 \left(\frac{6 - 2x}{3}\right)^2 dx$$

$$= \frac{1}{18} \int_0^3 (36 - 24x + 4x^2)\, dx$$

$$= \frac{1}{18} \left(36x - 12x^2 + \frac{4}{3}x^3\right)\Big|_0^3$$

$$= 2(3) - \frac{2}{3}(3)^2 + \frac{2}{(9)(3)}(3)^3 = 2$$

$$M_y = \int_0^3 xy\, dx = \int_0^3 x \left(\frac{6 - 2x}{3}\right) dx$$

$$= \frac{1}{3} \int_0^3 (6x - 2x^2)\, dx = \frac{1}{3}\left(3x^2 - \frac{2}{3}x^3\right)\Big|_0^3 = 3$$

$$\bar{x} = \frac{M_y}{A}, \quad \bar{y} = \frac{M_x}{A} \Rightarrow (\bar{x}, \bar{y}) = \boxed{\left(1, \frac{2}{3}\right)}$$

5. (10 points each) (a) Find the exact length of the polar curve $r = 5\sin\theta$, $0 \le \theta \le \pi/4$.

$$L = \int_0^{\pi/4} \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta$$

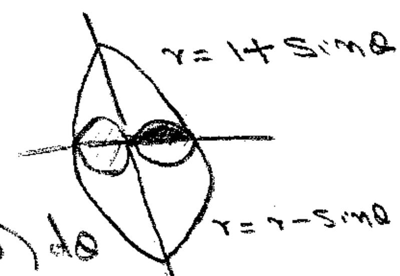
$$= \int_0^{\pi/4} \sqrt{25\sin^2\theta + 25\cos^2\theta}\, d\theta$$

$$= \int_0^{\pi/4} 5\sqrt{\sin^2\theta + \cos^2\theta}\, d\theta = \int_0^{\pi/4} 5\, d\theta$$

$$= 5\theta \Big|_0^{\pi/4} = \boxed{\frac{5}{4}\pi}$$

(b) Find the area of the region that lies inside the curves

$$r = 1 - \sin\theta \quad \text{and} \quad r = 1 + \sin\theta.$$



$$A = \cancel{4}^{2} \cdot \frac{1}{\cancel{2}} \int_0^{\pi/2} (1 - \sin\theta)^2\, d\theta$$

$$= 2\int_0^{\pi/2} (1 - 2\sin\theta + \sin^2\theta)\, d\theta$$

$$= 2\int_0^{\pi/2} \left(1 - 2\sin\theta + \frac{1}{2}(1 - \cos 2\theta)\right) d\theta$$

$$= 2\int_0^{\pi/2} \left(\frac{3}{2} - 2\sin\theta - \frac{1}{2}\cos 2\theta\right) d\theta$$

$$= 2\left(\frac{3}{2}\theta - 2\cos\theta - \frac{1}{4}\,\frac{\sin 2\theta}{2}\right)\Big|_0^{\pi/2}$$

$$= 2\left(\frac{3}{2}\,\frac{\pi}{2} - (-2(1))\right) = \boxed{\frac{3}{2}\pi + 4}$$

6. (10 points each) (a) Find the area of the region enclosed by the one loop of the curve $r = 3\sin 6\theta$.

$\sin 6\theta = 0$

$6\theta = 0, \pi \Rightarrow \theta = 0, \frac{\pi}{6}$

$$A = \frac{1}{2}\int_0^{\pi/6} (3\sin 6\theta)^2\, d\theta$$

$$= \frac{9}{2}\int_0^{\pi/6} \left(\frac{1}{2}(1-\cos(12\theta))\right) d\theta$$

$$= \frac{9}{4}\left(\theta - \frac{\sin(12\theta)}{12}\right)\Bigg|_0^{\pi/6}$$

$$= \frac{9}{4}\,\frac{\pi}{6} = \boxed{\frac{3}{8}\pi}$$

(b) Find the sum of the series $\displaystyle\sum_{n=0}^{\infty} \frac{1}{5^n}$ $= \dfrac{1}{1-\frac{1}{5}}$ (G.S.)

$$= \boxed{\frac{5}{4}}$$

7. (10 points each) (a) For what values of p is the series $\sum_{n=1}^{\infty} n^p \ln n$ convergent?

If $p \geq 0$, $\lim_{n\to\infty} n^p \ln n = \infty$, so series diverges.

If $p < 0$, $\frac{d}{dx}(x^p \ln x) < 0 \iff x > e^{-1/p}$, so series is decreasing on $[e^{-1/p}, \infty)$.

$$\int_1^{\infty} x^p \ln x \, dx = \lim_{t\to\infty} \left[\frac{x^{p+1}}{p+1} \ln x \Big|_1^t - \frac{1}{p+1} \int_1^t x^p \, dx \right] ; p \neq -1.$$

$$= \lim_{t\to\infty} \left[\frac{t^{p+1}}{p+1} \ln t - \frac{1}{(p+1)^2} (t^{p+1} - 1) \right]$$

is finite $\iff p+1 < 0 \iff p < -1$

Convergent for $\boxed{p < -1}$ (IT)

(b) Test the series $\sum_{n=1}^{\infty} \left(\frac{3n^3-2}{n^3+7} \right)^n$ for convergence or divergence.

$$|a_n|^{1/n} = \frac{3n^3-2}{n^3} \longrightarrow 3 > 1$$

$\boxed{\text{Divergent}}$ (Root Test)

8. (10 points each) (a) Find the radius of convergence of the series $\sum_{n=1}^{\infty} \frac{5^n x^n}{n^3}$.

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{5^{n+1} x^{n+1}}{(n+1)^3} \cdot \frac{n^3}{5^n x^n}\right|$$

$$= 5|x|\left(\frac{n}{n+1}\right)^3 \longrightarrow 5|x| < 1$$

$$\Leftrightarrow |x| < \frac{1}{5}$$

$$\boxed{R = \frac{1}{5}}$$ (Ratio Test)

(b) Determine whether the series $\sum_{n=1}^{\infty} \frac{3^{1/n}}{n}$ converges or diverges.

$$\sum_{n=1}^{\infty} \frac{3^{1/n}}{n} > \sum_{n=1}^{\infty} \frac{1^{1/n}}{n} = \sum_{n=1}^{\infty} \frac{1}{n}$$

$\boxed{\text{Divergent}}$ (CT).

9. (10 points each) (a) Test the following series for convergence or divergence

$$\frac{7}{2} - \frac{7 \cdot 12}{2 \cdot 5} + \frac{7 \cdot 12 \cdot 17}{2 \cdot 5 \cdot 8} - \frac{7 \cdot 12 \cdot 17 \cdot 22}{2 \cdot 5 \cdot 8 \cdot 11} + \ldots$$

$$= \sum_{n=1}^{\infty} (-1)^{n-1} \frac{7 \cdot 12 \cdots (5n+2)}{2 \cdot 5 \cdots (3n-1)}$$

$$\left|\frac{a_{n+1}}{a_n}\right| = \frac{7 \cdot 12 \cdots (5n+2)(5n+7)}{2 \cdot 5 \cdots (3n-1)(3n+2)} \cdot \frac{2 \cdot 5 \cdots (3n-1)}{7 \cdot 12 \cdots (5n+2)}$$

$$= \frac{5n+7}{3n+2} \longrightarrow \frac{5}{3} > 1$$

$\boxed{\text{Divergent}}$ (Ratio Test).

(b) Find the sum of the series $1 + \cos 2 + \frac{(\cos 2)^2}{2!} + \frac{(\cos 2)^3}{3!} + \ldots$.

$$= \sum_{n=0}^{\infty} \frac{(\cos 2)^n}{n!}$$

$$= \boxed{e^{\cos 2}}$$

10. (10 points each) (a) Test the following series for convergence or divergence

$$\sin^2(1) + \sin^2(1/2) + \sin^2(1/3) + \sin^2(1/4) + \ldots$$

$$= \sum_{n=1}^{\infty} \sin^2(1/n) \quad ; \quad b_n = \frac{1}{n^2}$$

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{1/n \to 0} \left(\frac{\sin(1/n)}{1/n}\right)^2 = 1^2 = 1 \neq 0, \neq \infty.$$

$\boxed{\text{Convergent}}$ (LCT).

(b) Test the series $\sum_{n=1}^{\infty} (\sqrt[n]{e} - 1)$ for convergence or divergence.

$$b_n = \frac{1}{n}.$$

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{e^{1/n} - 1}{1/n} \quad \left(\frac{0}{0}\right)$$

$$\stackrel{H}{=} \lim_{n\to\infty} \frac{e^{1/n} \cdot (-1/n^2)}{-1/n^2} = e^0 = 1$$

$$\neq 0, \neq \infty.$$

$\boxed{\text{Divergent}}$ (LCT).

$$\int \sec u \, du = \ln|\sec u + \tan u| + c$$

$$\int \csc u \, du = \ln|\csc u - \cot u| + c$$

$$\int \tan u \, du = \ln|\sec u| + c \qquad \int \cot u \, du = \ln|\sin u| + c$$

Centroid for the region trapped between $y = f(x)$, $y = g(x)$, $a \le x \le b$, (with $\rho = 1$)
$M_x = \frac{1}{2}\int_a^b f(x)^2 - g(x)^2 \, dx$, $M_y = \int_a^b x(f(x) - g(x)) \, dx$

Maclaurin Series: $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$, $(-\infty, \infty)$

$\ln(1+x) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}x^n}{n}$, $(-1, 1]$

$\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}$, $(-\infty, \infty)$

$\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}$, $(-\infty, \infty)$

$\tan^{-1} x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2n+1}$, $(-1, 1]$