Name______________________________ Instructor_______________________________
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Analytic Geometry and Calculus I
Exam 1
Summer 2009

Show all work for full credit. You may use a calculator, but no books or notes. The point for each problem is given in the left of the problem.

1. Sketch the graph of the following function and determine whether it is continuous or not.

$$f(x) = \begin{cases} 2x+1 \ if \ \ x<0 \\ x^2+1 \quad \ \ if \ \ 0 \le x \le 1 \\ 4-x \ \ if \ \ x>1 \end{cases}$$

Solution: The function f(x) has three polynomial parts which are divided by domain points x=0 and x=1. Each polynomial part is continuous. So, we need to check whether the function is continuous at those two points.
At x=0:

$$\lim_{x\to 0^-} f(x) = \lim_{x\to 0^-}(2x+1) = 1 \ and \ \lim_{x\to 0^+} f(x) = \lim_{x\to 0^+}(x^2+1) = 1$$

$$So, \lim_{x\to 0^-} f(x) = \lim_{x\to 0^+} f(x) \ implies \ f(x) \ is \ continuous \ at \ \ x=0$$

At x=1:

$$\lim_{x\to 1^+} f(x) = \lim_{x\to 1^+}(4-x) = 3 \ and \ \lim_{x\to 1^-} f(x) = \lim_{x\to 1^-}(x^2+1) = 2$$

$$So, \lim_{x\to 1^+} f(x) \ne \lim_{x\to 1^-} f(x) \ implies \ f(x) \ is \ not \ continuous \ at \ \ x=1$$

2. Check whether the following function is one-to-one or not. If yes, find the inverse function of it.

$$f(x) = x^3+1$$

Solution: This function is vertical shifting of graph of function $y=x^3$, which is one-to-one, so is one-to-one. Then,

$$y = x^3+1 \Rightarrow x = \sqrt[3]{y-1} \Rightarrow f^{-1}(x) = \sqrt[3]{x-1}$$

3. Find an equation of the tangent line to the curve $y=2x^3+x^2-3$ at the point x=1.

Solution:

$$\frac{dy}{dx} = 6x^2+2x \ so \ m = \left[\frac{dy}{dx}\right]_{x=1} = 8.$$

$$The \ equation \ to \ the \ line \ having \ slope \ 8 \ and \ a \ point \ (1,0) \ is,$$

$$y-0 = 8(x-1) \ i.e. 8x-y = 8$$

4. Find the limits of:

a) $\lim_{x\to 3}\frac{x^2-9}{x-3}$ b) $\lim_{x\to 4}\frac{\sqrt{x^3}-2x}{x-4}$ c) $\lim_{x\to\infty}\frac{4x^2-9x+7}{3x^2-5x+2}$

Solution:

$$a)\ \lim_{x\to 3}\frac{x^2-9}{x-3}=\lim_{x\to 3}\frac{(x+3)(x-3)}{x-3}=\lim_{x\to 3}(x+3)=6$$

$$b)\lim_{x\to 4}\frac{\sqrt{x^3}-2x}{x-4}=\lim_{x\to 4}\frac{x\sqrt{x}-2x}{x-4}=\lim_{x\to 4}\frac{x(\sqrt{x}-2)}{(\sqrt{x}+2)(\sqrt{x}-2)}=\lim_{x\to 4}\frac{x}{(\sqrt{x}+2)}=\frac{2}{4}=\frac{1}{2}$$

$$c)\lim_{x\to\infty}\frac{4x^2-9x+7}{3x^2-5x+2}\lim_{x\to\infty}\frac{4-9/x+7/x^2}{3-5/x+2/x^2}=\frac{4}{3}$$

5. Find the derivatives of each of function.

a) $f(x)=x^3+2x^{(3/2)}+4$ b) $y=x^3e^x+\ln x$ c) $y=\frac{x^3}{3x^2+2}$

d) $y=x^3\sin(2x-1)+e^{3x}\ln(x)$ e) $y=\sin(\ln(3x^2+2))$

Solution:

$$a) f'(x)=3x^2+2(3/2)x^{(3/2)-1}=3x^2+3\sqrt{x}=3(x^2+\sqrt{x})$$

$$b)\frac{dy}{dx}=\frac{d(x^3e^x+\ln x)}{dx}=\frac{d(x^3e^x)}{dx}+\frac{d(\ln x)}{dx}=(3x^2e^x+x^3e^x)+\frac{1}{x}=3x^2e^x+x^3e^x+\frac{1}{x}$$

$$c)\frac{dy}{dx}=\frac{d}{dx}\left(\frac{x^3}{3x^2+2}\right)=\frac{(3x^2+2)(3x^2)-(x^3)(6x)}{(3x^2+2)^2}=\frac{3x^2(x^2+2)}{(3x^2+2)^2}$$

$$d)\frac{dy}{dx}=(3x^2\sin(2x-1)+x^3\cos(2x-1)(2))+(3e^{3x}\ln(x)+e^{3x}/x)$$

$$=3x^2\sin(2x-1)+2x^3\sin(2x-1)+3e^{3x}\ln(x)+e^{3x}/x$$

$$e)\frac{dy}{dx}=\cos(\ln(3x^2+2))\left(\frac{1}{3x^2+2}\right)(6x)=\left(\frac{6x}{3x^2+2}\right)\cos(\ln(3x^2+2))$$

6) Find dy/dx of implicit function $y^2+xy+x^2=3$.

Solution:

$$2y\frac{dy}{dx}+\left(y+x\frac{dy}{dx}\right)+2x=0\Rightarrow(2y+x)\frac{dy}{dx}=-(2x+y)\Rightarrow\frac{dy}{dx}=\frac{-(2x+y)}{2y+x}=-\frac{2x+y}{2y+x}$$

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Analytic Geometry and Calculus I
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1. Find the f', f'' of the following functions:

a) $f(x) = x^3 + xe^x + 4$

b) $f(x) = x^3 \sin(x^2)$

Solution:

$a) f'(x) = 3x^2 + (e^x + xe^x) = 3x^2 + e^x + xe^x \quad and$

$f''(x) = 6x + e^x + (e^x + xe^x) = 6x + 2e^x + xe^x$

$b) f'(x) = 3x^2 \sin(x^2) + x^3 \cos(x^2)(2x) = 3x^2 \sin(x^2) + 2x^4 \cos(x^2)$

$f''(x) = 3[2x\sin(x^2) + x^2 \cos(x^2)(2x)] + 2[4x^3 \cos(x^2) + x^4 \sin(x^2)(2x)]$

$= 6x\sin(x^2) + 14x^3 \cos(x^2) + 4x^5 \sin(x^2)$

2. Find an equation of tangent line to the curve $x^2+xy+y^2=3$ at (1,1).

Solution:

Differentiation both sides w.r.t x, we get,

$$2x + \left(y + x\frac{dy}{dx} \right) + 2y\frac{dy}{dx} = 0 \Rightarrow \frac{dy}{dx} = -\frac{2x + y}{x + 2y} \Rightarrow m = \left[\frac{dy}{dx} \right]_{(x,y)=(1,1)} = -\frac{3}{3} = -1.$$

So the tangent line is

$y - 1 = (-1)(x - 1) \Rightarrow x + y + 2 = 0$

3. A balloon is being inflating using a pump of capacity 10 cubic inches per second. If the balloon is growing spherically in size, find the rate at which the radius is increasing when the radius is 6 inches?

Solution:

$$Given\ that\ \frac{dV}{dt} = 10\,in^3/s\ at\ r = 6\,in.$$

$$We\ know\ V = \frac{4\pi r^3}{3}.\ So$$

$$\frac{dV}{dt} = 4\pi r^2 \frac{dr}{dt} \Rightarrow \frac{dr}{dt} = \left(\frac{1}{4\pi r^2} \right)\left(\frac{dV}{dt} \right) = \frac{1}{144\pi}(10) = \frac{5}{72\pi}\,in/s$$

4. Approximate $\sqrt[4]{15}$ using as:

a) Find the equation of the tangent line to the curve $f(x)=\sqrt[4]{x}$ at x=16 and use this to approximate $\sqrt[4]{15}$.

Solution:

$$\frac{dy}{dx}=\frac{1}{4}x^{-(3/4)} \Rightarrow m=\left(\frac{dy}{dx}\right)_{x=16}=\frac{1}{32} \textit{ and at } (16,2). \textit{So the } \tan \textit{gent line is}$$

$$y-2=\left(\frac{1}{32}\right)(x-16). \textit{ Then } y=2+\left(\frac{1}{32}\right)(x-16). \textit{ At } x=15, \textit{ we get}$$

$$f(15)=\sqrt[4]{15}=2+\left(\frac{1}{32}\right)(15-16)=2-\frac{1}{32}=\frac{63}{32}\approx 1.96875 \textit{ which is close to the}$$

$$\textit{calculator value } 1.96789$$

b) Apply Newton's Method to $f(x)=x^4-15$ and compute $x_2=x_1-\frac{f(x_1)}{f'(x_1)}$ at $x_1=2$.

Solution:

$$x_2=x_1-\frac{f(x_1)}{f'(x_1)}=x_1-\frac{x_1^{\ 4}-15}{4x_1^{\ 3}}$$

$$\Rightarrow x_2\,|_{x_1=2}=2-\frac{2^4-15}{4(2^3)}=2-\frac{1}{32} \textit{ which is as part } a).$$

5. Find the y' and y'' of the function $y=a^{\sin(x)}$.

Solution:

$$y'=a^{\sin(x)}\ln a \textit{ and } y''=a^{\sin(x)}(\ln a)^2$$

6) The position of a moving particle is given by $x=t^3-4t^2+3t+6$ meters, where t is in seconds. Find the velocity and acceleration of the object when t=2s.

Solution:

$$\textit{velocity } v=\frac{dx}{dt}=3t^2-8t+3 \textit{ so } v\,|_{t=2s}=-1\,m/s \textit{ and}$$

$$v=0 \Rightarrow t=\frac{8\pm\sqrt{64-36}}{6}=\frac{8\pm 2\sqrt{7}}{6}=\frac{1}{3}[4+\sqrt{7}]s \textit{ or } \frac{1}{3}[4-\sqrt{7}]s$$

$$\textit{acceleration } a=\frac{d^2x}{dt^2}=6t-8 \Rightarrow a\,|_{t=2s}=4m/s\,.$$

7. a) Find the maximum and minimum of $f(x)=x^3-6x^2+9x+4$ on [1,4]. Also identify the absolute maximum and minimum if exists.

Solution:

$$f'(x)=3x^2-12x+9. \textit{Setting } 3x^2-12x+9=0 \textit{ we get } x=1 \textit{ or } 3$$

x	f(x)	Result
1	8	Absolute Maximum
3	4	Absolute minimum
4	8	Absolute Maximum

b) Using the information from part a), draw the graph of the function on part a) and find the point of inflection.

Solution:

$f''(x) = 6x - 12. Setting\ 6x - 12 = 0\ we\ get\ x = 2. So\ the\ po\text{int}\ of\ \text{inf}\ lection\ is\ x = 2.$

8. Evaluate the following limits:

a) $\lim_{x\to 0}\frac{\cos(x)-1}{x^2}$ b) $\lim_{x\to 0} x^{\sin(x)}$

Solution:

$$a)\lim_{x\to 0}\frac{\cos(x)-1}{x^2} = \lim_{x\to 0}\frac{-\sin(x)}{2x} = \lim_{x\to 0}\frac{-\cos(x)}{2} = -\frac{1}{2}$$

$$\text{a)}\,b)\lim_{x\to 0} x^{\sin(x)} = \lim_{x\to 0} e^{\sin(x)\ln x} = e^{\lim_{x\to 0}\sin(x)\ln x} = e^{\lim_{x\to 0}\sin(x)\ln x} = e^{\lim_{x\to 0}\frac{\ln x}{\csc(x)}} = e^{\lim_{x\to 0}\frac{1/x}{-\csc x\cot(x)}}$$

$$= e^{-\lim_{x\to 0}\frac{\tan(x)}{x}(\sin x)} = e^{-(1)(0)} = 1$$

9. Find the anti-derivative (integral) of

a) $3x^2 + 2$ b) $\frac{x+1}{x^2+2x+3}$

Solution:

$a)F(x) = x^3 + 2x + c$

$$b)Consider\ \ln(x^2+2x+3). Then\frac{d[\ln(x^2+2x+3)]}{dx} = \frac{2x+2}{x^2+2x+3} = \frac{2(x+1)}{x^2+2x+3}. That\ is$$

$$\frac{d}{dx}\left[\frac{1}{2}\ln(x^2+2x+3)\right] = \frac{(x+1)}{x^2+2x+3} \Rightarrow F(x) = \frac{1}{2}\ln(x^2+2x+3) + C$$