

“Easy” Trigonometric Equations

In this note we outline the method of solving equations the form:

$$\sin t = \text{number} \tag{1}$$

$$\cos t = \text{number} \tag{2}$$

$$\tan t = \text{number}. \tag{3}$$

Before we proceed, we need to recal several key facts (items A, B, C below).

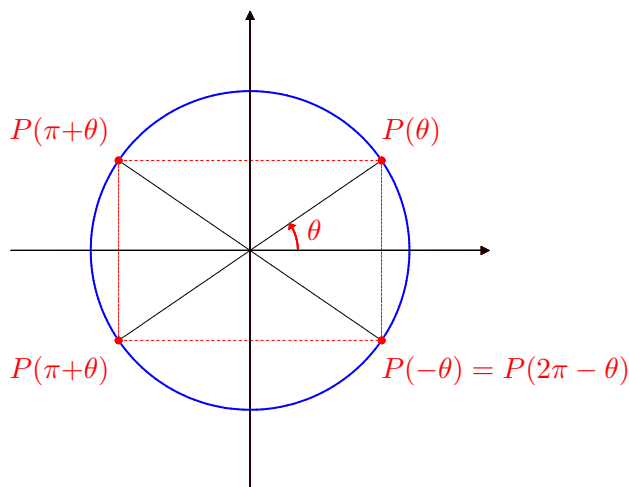
A. Features of sin, cos, and tan. These are collected in the table below.

Trigonometric Function	Fundamental Interval	Period	Graph over the Fundamental Interval
$\sin t$	$[0, 2\pi)$	2π	
$\cos t$	$[0, 2\pi)$	2π	
$\tan t$	$(-\frac{\pi}{2}, \frac{\pi}{2})$	π	

B. The formula “packages.” Except for the formulas for negatives, the formulas below are not explicitly identified in the textbook. These formulas are nevertheless important.

Formulas for Negatives & “Back from 2π ” Formulas	“Add π ” Formulas	Supplement Formulas
$\sin(-\theta) = \sin(2\pi - \theta) = -\sin \theta$ $\cos(-\theta) = \cos(2\pi - \theta) = \cos \theta$ $\tan(-\theta) = \tan(2\pi - \theta) = -\tan \theta$	$\sin(\pi + \theta) = -\sin \theta$ $\cos(\pi + \theta) = -\cos \theta$ $\tan(\pi + \theta) = \tan \theta$	$\sin(\pi - \theta) = \sin \theta$ $\cos(\pi - \theta) = -\cos \theta$ $\tan(\pi - \theta) = -\tan \theta$

TIP: The easiest way to remember these formulas is to use the picture to the right. The points $P(-\theta) = P(2\pi - \theta)$, $P(\pi + \theta)$, and $P(\pi - \theta)$ have the pretty much same coordinates as $P(\theta)$, *except for some sign adjustments*. (Note that this picture illustrates the case when θ is in the *first quadrant*.)



C. “Familiar” Values. For future reference, here are some “familiar” values for trigonometric functions:

θ	0 (0°)	$\frac{\pi}{6}$ (30°)	$\frac{\pi}{4}$ (45°)	$\frac{\pi}{3}$ (60°)	$\frac{\pi}{2}$ (90°)	π (180°)
$\sin \theta$	0	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$	1	0
$\cos \theta$	1	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0	-1
$\tan \theta$	0	$\frac{\sqrt{3}}{3}$	1	$\sqrt{3}$	undefined	0

D. Finding solutions in the fundamental interval. Let us consider the equations (1)-(3), where the unknown t is in the *fundamental interval*. We refer to one such equation as:

$$\text{func } t = \text{number.}$$

To solve such an equation in the fundamental interval (see §A for the specifics), we proceed as follows:

1. Solve the equation

$$\text{func } \theta = \text{abs number,}$$

where “**abs number**” is the *absolute value* of **number**, and θ is in the *first quadrant*, that is, $0 \leq \theta \leq \frac{\pi}{2}$.

2. Pick t from the list (use the “packages” from §B to decide which values work):

$$-\theta, \theta, \pi - \theta, \pi + \theta, 2\pi - \theta.$$

Depending on the trigonometric function involved in the equation, the above list can be shortened as indicated below.

- (i) If **func** is **sin** or **cos**, then $-\theta$ should be omitted from the list. Moreover, in this case the equation typically has **two** solutions in the fundamental interval.
- (ii) If **func** is **tan**, the list has only two numbers: $-\theta, \theta$. Moreover, in this case the equation has only **one** solution in the fundamental interval.

TIP: If θ is in the *first quadrant*, instead of using the formula “packages,” one can decide which values of t should be selected, based on the *quadrant* information (which gives us the *sign* information) summarized below:

QUADR. I	QUADR. II	QUADR. III	QUADR. IV
θ	$\pi - \theta$	$\pi + \theta$	$-\theta, 2\pi - \theta$

EXAMPLE 1. Solve the equation $\tan t = -\sqrt{3}$ in the fundamental interval.

Solution. We start off by solving the equation $\tan \theta = \sqrt{3}$ in the first quadrant. Using the “familiar” values (see §C) we find $\theta = \frac{\pi}{3}$. The solution(s) t should be then picked from the following list:

- $\frac{\pi}{3}$ (QUADR. I),
- $-\frac{\pi}{3}$ (QUADR. IV).

The solution is then $t = -\frac{\pi}{3}$.

EXAMPLE 2. Solve the equation $\sin t = -\frac{\sqrt{2}}{2}$ in the fundamental interval.

Solution. We start off by solving the equation $\sin \theta = \frac{\sqrt{2}}{2}$ in the first quadrant. Using the “familiar” values (see §C) we find $\theta = \frac{\pi}{4}$. The solution(s) t should be then picked from the following list:

- $\frac{\pi}{4}$ (QUADR. I),
- $\pi - \frac{\pi}{4}$ (QUADR. II),
- $\pi + \frac{\pi}{4}$ (QUADR. III),
- $2\pi - \frac{\pi}{4}$ (QUADR. IV).

The solutions are then $t_1 = \pi + \frac{\pi}{4} = \frac{5\pi}{4}$ and $t_2 = 2\pi - \frac{\pi}{4} = \frac{7\pi}{4}$.

E. Finding all solutions of the equation. We now consider the problem of finding all the solutions of an equation of the form:

$$\text{func } t = \text{number.}$$

To solve this problem, one proceeds as follows:

1. Solve the equation in the fundamental interval. Call these solutions the *basic* ones.
2. Use each basic solution to generate a *family of solutions*, constructed like

$$t = \text{basic sol.} + n \cdot \text{Period}, \quad n \text{ integer.}$$

The two examples below follow the examples 1, 2 above.

EXAMPLE 3. Find *all* solutions of the equation $\tan t = -\sqrt{3}$.

Solution. By the discussion in Example 1, the basic solution is $t_1 = -\frac{\pi}{3}$. The period of \tan is π , so all solutions are given as

$$t = -\frac{\pi}{3} + n\pi, \quad n \text{ integer.}$$

EXAMPLE 4. Find *all* solutions of the equation $\sin t = -\frac{\sqrt{2}}{2}$.

Solution. By the discussion in Example 2, the basic solutions are $t_1 = \frac{5\pi}{4}$ and $t_2 = \frac{7\pi}{4}$. The period of \sin is 2π , so all solutions are given by two families:

$$\begin{aligned}t &= \frac{5\pi}{4} + 2n\pi, & n \text{ integer,} \\t &= \frac{7\pi}{4} + 2n\pi, & n \text{ integer.}\end{aligned}$$

F. Finding solutions in a given interval. To solve the equation

$$\text{func } t = \text{number}$$

in a *given* interval, we can employ one of the following methods.

Method I. (i) Solve in the fundamental interval.

(ii) Find *all* solutions (generate the family/families as explained in §E).

(iii) From each family identify which values of n yield solutions in the prescribed interval.

(iv) Go back to (ii) and compute.

Method II. (i) Sketch the graph, count and identify the desired solutions.

(ii) Indicate how the desired solutions relate to the basic solutions (in the fundamental interval).

(iii) Solve in the fundamental interval.

(iv) Go back to (ii) and compute.

Below we illustrate each method on the examples 1(3) and 2(4) above.

EXAMPLE 5. Solve the equation $\tan t = -\sqrt{3}$ in the interval $[2\pi, 5\pi]$.

Solution (based on Method I). As discussed in Example 3, *all* solutions of the equation are constructed as

$$t = -\frac{\pi}{3} + n\pi, \quad n \text{ integer,}$$

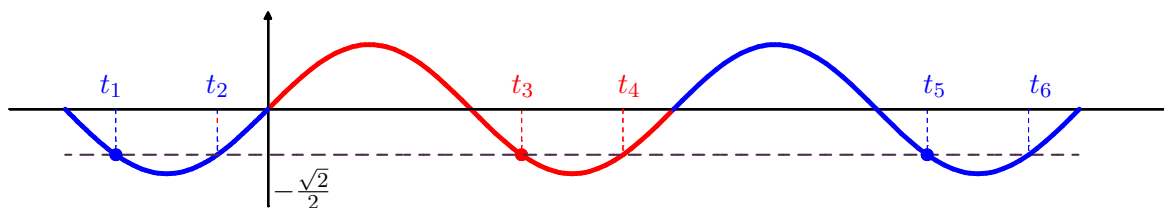
so all we need to find are the values of n which produce t 's in the interval $[2\pi, 5\pi]$. These

values are $n = 3, 4, 5$, so the solutions are $t_1 = -\frac{\pi}{3} + 3\pi = \frac{8\pi}{3}$, $t_2 = -\frac{\pi}{3} + 4\pi = \frac{11\pi}{3}$, and

$$t_3 = -\frac{\pi}{3} + 5\pi = \frac{14\pi}{3}.$$

EXAMPLE 6. Solve the equation $\sin t = -\frac{\sqrt{2}}{2}$ in the interval $[-\pi, 4\pi]$.

Solution (based on Method II). The picture below shows the graph of $\sin t$ over the interval $[-\pi, 4\pi]$. (The part of the picture that corresponds to the fundamental interval is drawn in red; the rest is drawn in blue.)



Upon inspecting this graph, we find that there are **six** solutions. Using the labeling from the picture, the solutions in the fundamental interval are t_3 and t_4 , while the remaining solutions are: $t_1 = t_3 - 2\pi$, $t_2 = t_4 - 2\pi$, $t_5 = t_3 + 2\pi$, and $t_6 = t_4 + 2\pi$.

As discussed in Example 2, we know that $t_3 = \frac{5\pi}{4}$ and $t_4 = \frac{7\pi}{4}$. Consequently, the other solutions are:

- $t_1 = \frac{5\pi}{4} - 2\pi = -\frac{3\pi}{4}$;
- $t_2 = \frac{7\pi}{4} - 2\pi = -\frac{\pi}{4}$;
- $t_5 = \frac{5\pi}{4} + 2\pi = \frac{13\pi}{4}$;
- $t_6 = \frac{7\pi}{4} + 2\pi = \frac{15\pi}{4}$.