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Matrix Multiplication Studio

A matrix is a rectangular array of numbers. The "shape" of a matrix is the number of rows by the number of columns. For example

$$\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \text{ is a } 2\times 2 \text{ matrix}$$

$$\begin{pmatrix} 3 & -4 & 0 \\ 2 & 5 & 1 \end{pmatrix} \text{ is a } 2\times 3 \text{ matrix}$$

$$\begin{pmatrix} 3 \\ -2 \\ 0 \\ 7 \end{pmatrix} \text{ is a } 4\times 1 \text{ matrix}$$

In section 5.3, the book defines various arithmetic operations on matrices. In this lab, we will consider specifically matrix multiplication. This technique is useful in modeling a wide variety of situations (if you see the term Markov chains, then matrices are lurking), and are also vital to computer graphics. Understanding matrix multiplication is also key to developing a notion of matrix division – which is the technique we will use in the final studio to see how to get computers to solve systems of linear equations for you.

The product of two matrices is another matrix. You compute the element in the i^{th} row and the j^{th} column of the product matrix by taking the i^{th} row of the left factor, the j^{th} column of the right factor, multiplying matching entries, and then totaling up the products. Don't worry if you can't figure out that last sentence. No one ever does until they see an example. To multiply $\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \times \begin{pmatrix} -3 & 5 \\ 7 & -2 \end{pmatrix}$ we first take the first row of the left matrix, $\begin{pmatrix} 1 & 2 \end{pmatrix}$, and the first column of the right matrix, $\begin{pmatrix} -3 \\ 7 \end{pmatrix}$, multiply the corresponding terms, 1 ×(–3) = –3, and 2 × 7 = 14, and total these to get – 3 + 14 = 11. This becomes the entry in the 1^{st} row and 1^{st} column of the product matrix. Working out the other terms in a similar fashion gives the final result

$$\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \times \begin{pmatrix} -3 & 5 \\ 7 & -2 \end{pmatrix} = \begin{pmatrix} 1\times(-3)+2\times 7 & 1\times 5+2\times(-2) \\ 3\times(-3)+4\times 7 & 3\times 5+4\times(-2) \end{pmatrix} = \begin{pmatrix} 11 & 1 \\ 19 & 7 \end{pmatrix}.$$

There are a couple of complications in multiplying matrices that don't come up in multiplying real (or complex) numbers. One is that you can't multiply just any old pair of matrices. In order to get the terms to match up, you need to have the same number of columns on the left as you have rows on the right. In terms of shape, you can multiply a $m \times n$ matrix only by an $n \times r$ matrix, where the middle terms of the shape are equal. Another issue is that the rules for multiplication depend on which matrix is on the left and

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which is on the right, since you take rows from the left and columns from the right. That means that if A and B are two matrices, you may have $AB \neq BA$. Indeed, multiplying matrices in different orders will usually produce different answers, if the multiplication is defined at all. For example, you can multiply a 2×3 matrix by a 3×4 matrix in that order, but if you reverse them than you can't multiply the 3×4 matrix by the 2×3 matrix since the rows with 4 elements on the left will no longer match up with columns of 2 elements on the right.

Multiply the following (if possible):

1(a) $$\begin{pmatrix} 3 & -2 \\ 4 & 11 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ -3 & 5 \end{pmatrix} =$$

1(b) $$\begin{pmatrix} 1 & 0 \\ -3 & 5 \end{pmatrix}\begin{pmatrix} 3 & -2 \\ 4 & 11 \end{pmatrix} =$$

1(c) $$\begin{pmatrix} 3 & -1 & 2 \\ 2 & 0 & 5 \end{pmatrix}\begin{pmatrix} 7 & 0 & -2 \\ 1 & 3 & -3 \\ 2 & 4 & 5 \end{pmatrix} =$$

1(d) $$\begin{pmatrix} 7 & 0 & -2 \\ 1 & 3 & -3 \\ 2 & 4 & 5 \end{pmatrix}\begin{pmatrix} 3 & -1 & 2 \\ 2 & 0 & 5 \end{pmatrix} =$$

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We will have more practice with matrix multiplication on the next online assignment. For this studio, we will look at how matrix multiplication can be applied. As an example of how matrix multiplication can be used in modeling, we consider a simple macroeconomic/sociological model of economic classes. It is well established that an important factor in the economic class a person attains is the class in which they begin. The U.S. has historically had much higher rates of class mobility than most societies, though there are now claims that class mobility is decreasing. Suppose that a child born into the lower class has a 40% chance of growing up to be in the lower class and a 50% chance of growing up to be in the middle class, and other probabilities of attaining different economic classes as given in the table below.

		Parent's Economic Class		
		Low	*Middle*	*Upper*
Child's	*Low*	40.0%	10.0%	5.0%
Econ	*Middle*	50.0%	70.0%	50.0%
Class	*Upper*	10.0%	20.0%	45.0%
	Total	100.0%	100.0%	100.0%

These numbers are not real data, but the technique we will use to analyze the situation is important in many real applications. Suppose the population is currently 25% lower class, 65% middle class, and 10% upper class. What will we expect the distribution of wealth to look like in the next generation (assuming all classes have the same number of children per capita)? Well 40% of the children of the 25% lower class parents will end up in the lower class, which is $0.4\times0.25 = 10\%$ of the population. 10% of the children of the 65% middle class parents will end up in the lower class, which is $0.1\times0.65 = 6.5\%$. And 5% of the children of the 10% of upper class parents will end up in the lower class, which is $0.05\times0.1 = 0.5\%$. All total we will have $10\% + 6.5\% + 0.5\% = 17\%$ of people in the lower economic class in the next generation. We could then repeat these calculations to find the percentage of middle class and upper class people in the next generation as well. But these calculations are just exactly carrying out a matrix multiplication, and it is easiest to write them out in matrix form.

$$\begin{pmatrix} 0.4 & 0.1 & 0.05 \\ 0.5 & 0.7 & 0.5 \\ 0.1 & 0.2 & 0.45 \end{pmatrix}\begin{pmatrix} 0.25 \\ 0.65 \\ 0.1 \end{pmatrix} = \begin{pmatrix} 0.4\times0.25+0.1\times0.65+0.05\times0.1 \\ 0.5\times0.25+0.7\times0.65+0.5\times0.1 \\ 0.1\times0.25+0.2\times0.65+0.45\times0.1 \end{pmatrix} = \begin{pmatrix} 0.17 \\ 0.63 \\ 0.2 \end{pmatrix}$$

Observe that the top row of the matrix multiplication is exactly the computation we went through in the previous paragraph to compute the percentage in the lower class in the next generation. Similarly, the next generation will have 63% in the middle class and 20% in the upper class. Using a spreadsheet will let us automate these calculations and rapidly see how varying rates of mobility will affect long term mix of different economic classes.

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A. Open a new sheet on your spreadsheet. First we will enter the data on how economic class changes from generation to generation. Since we have a bunch of numbers here, we will spend a moment to clearly identify which number means what on the spreadsheet (which will help us keep things straight when we start changing numbers later). Click and drag to select cells C1..E1. Right-click on the selected cells and choose "Format Cells…" from the dialog box. Click on the alignment tab and then check the "Merge cells" box. Select OK to close the dialog box and produce a single cell spanning the three columns. Now enter "Parent's Economic Class" in the merged cell (I found it nicer if I centered the text and put it in bold-face). Next enter "Low" in cell C2, "Middle" in cell D2, and "High" in cell E2 (I italicized these labels). Now enter the labels "Child's" in cell A3, "Econ" in cell A4, and "Class" in cell A5 (I bolded these, but I've never found merging cells vertically or twisting the text to run vertically to be helpful in reading a table). Finally enter the labels "Low" in cell B3, "Middle" in cell B4, and "High" in cell B5 (I italicized these to match the labels for the parent's class).

B. Now that we have set up the labels, fill in the data values given above into the cells C3..E5. Note that since you are entering percents, you must either type in the percent sign (e.g. 40%) or enter the decimal equivalent (e.g. 0.4). As a check, go to cell C7 and enter the formula "=sum(c3:c5)" The sum of the possible outcomes for a child whose parent's were in the lower class should total to 100%. If it doesn't, you've entered something wrong. Label this check value with "Total" in cell B7, then click and drag to pull the formula from cell C7 to cells D7 and E7 to be sure you have those columns set up correctly as well. If you have been entering decimals, it will be easier to read the values if you format the cells to display percentages. Select the cells to be formatted. Since we will want this formatting to extend to the right in a minute, it will be easiest to select whole rows. Click on the 3 that marks row 3 at the very left edge of the screen. Be sure to click in the middle of the 3 (where the pointer forms a black arrow pointing to the right) and not at the edge of the row (where the pointer forms a double arrow pointing up and down with a line between them). Then drag the pointer down to select all of rows 3 through 7. Now right-click in the selected region and choose "Format Cells…" from the dialog box. Select the Number tab and then select the Category "Percentage." **Set the number of decimal places to 1**, and then click OK. All the numerical values in the row will now be listed as percentages. Your spreadsheet should now look like the table listed towards the top of the previous page.

C. Now that we have the mobility data entered into what we will call the "transition matrix," we will enter the initial values for how many people are in each class, and then compute how these values change through the generations. Select cells G1..S1 and merge them to form one long cell, then enter the label "Generation" into this merged cell (you will want to center it). Enter 0 in cell G2, 1 in cell H2, on through 12 in cell S2. Now enter the initial populations, 25% in cell G3, 65% in cell G4, and 10% in cell G5. Also enter the formula "=sum(g3..g5)" into cell G7 to check you have 100% of the population accounted for.

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D. Next we want to multiply the transition matrix by the initial population. Click and drag to select cells H3..H5. Enter the formula "=mmult($C$3:$E$5,G3:G5)" and press Ctrl-Shift-Enter to enter the matrix formula. Using Ctrl-Shift-Enter means the formula will be computed for the whole rectangular array of cells selected. You can click and drag the total formula from cell G7 over to cell H7 to check the sum of the different populations is still 100%. Your spreadsheet should now look like this.

		Parent's Economic Class				
		Low	*Middle*	*Upper*	**0**	**1**
Child's	*Low*	40.0%	10.0%	5.0%	25.0%	17.0%
Econ	*Middle*	50.0%	70.0%	50.0%	65.0%	63.0%
Class	*Upper*	10.0%	20.0%	45.0%	10.0%	20.0%
	Total	100.0%	100.0%	100.0%	100.0%	100.0%

E. Now of course, carrying out a single matrix multiplication is usually easier by hand (or with a calculator) than with a spreadsheet. The advantage of a spreadsheet is that you can now repeat this calculation many times very easily. For instance, to compute what the situation will be like in the next generation, you would multiply the values for generation 1 by the transition matrix one more time. We can do this rapidly now that the spreadsheet is properly set up. Click and drag to select cells H3..H7. Then click on the cross in the lower right of the selection and drag it over to row S to compute how economic classes will develop over a dozen generations. Now that we have the spreadsheet, we can also look at how changing different values will affect things just by typing new values into the correct cells.

2. Notice that the percentages in each economic class settle down and stop changing after half-a-dozen generations or so. *Suppose you were able to wave a magic wand and move 100% of the population in generation 0 into the upper class. How would this affect the percentages over the long run?*

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3. Suppose a major economic dislocation (think Great Depression) changes things so 70% of the population is low, 25% middle, and 5% high. *How would this affect the percentages over the long run?*

4. Suppose society changed so that it became easier for children to escape poverty into the middle class. Change the percentage of children of low income parents who end up in the lower class to 35% (in cell C3), and increase the percent who become middle class to 55% (in cell C4). *Does this change the long term percentages in each class?* Note that we just changed transitions affecting low and middle class. *Does this change just affect just the long-term low and middle class percentages?*

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5. Suppose you could make changes to society that will move 5% out of one box of the transition matrix down one box (as in the last problem you moved 5% from the top left cell down to the cell below). *Moving 5% between which pair of cells (with one right on top of the other) will cause the greatest long-term decrease in poverty?*

Since the transition matrix was not real data, obviously you shouldn't make too much about the results you got here, though it is true that relatively small changes in the transition matrix can cause large long-term changes, while changes in the current population will fade away with time. But the general technique is important for studying many situations where you are studying how something moves between several different states.