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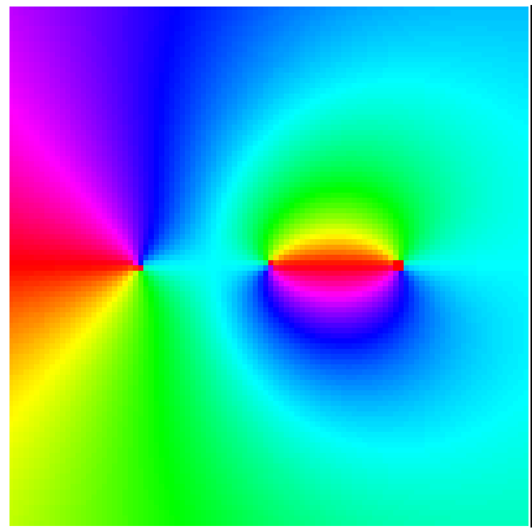
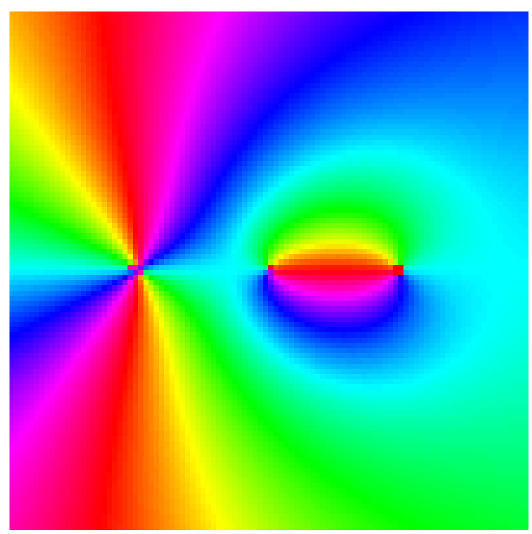
Studio College Algebra
Fundamental Theorem of Algebra Extra Credit

The fundamental theorem of algebra is given on p. 417 of the text. It says any polynomial has at least one complex zero (note that real numbers are also complex numbers, just with 0 imaginary part). No justification is given for the result, but immediately afterward the book notes that once you've proven the fundamental theorem, you can extend it from showing there is a zero to showing that an n^{th} degree polynomial has exactly n zeros, if you include complex zeros and count with multiplicity. While we won't be able to give a precise proof, in this extra credit assignment you will be able to find some patterns that can be used to explain why the fundamental theorem is true.

Step 1: The Principle of the Argument



The key fact we will need to justify the Fundamental Theorem of Algebra, is a result called the Principle of the Argument (the technical mathematical term for the angle of a complex number is the "argument" – so argument in this context refers to the color of the point in our graphs since the color represents the angle). This principle addresses how the arguments behave around the boundry of a region. Consider the top view of the graph of $\frac{z^2+z}{z-1}$. This function has roots at $z = 0$ and $z = -1$, and a pole at $z = 1$. You can see these in the graph at the right as the three points where the colors come together. If you "follow the rainbow" from red to yellow to green to blue to purple (and back to red), the colors move counterclockwise around the roots and clockwise around the pole. That's what we discovered in studio. But now, instead of looking at how the colors circle around the roots and poles, we will look at how they circle around the edge of the graphed region. In this case, if we follow the colors around the edge of the graph, they go through the cycle exactly once counterclockwise (starting with red at the "west" edge of the graph, going through yellow at the "southwest," green at the "south," blue all the way from the "southeast to the "northeast," on through indigo (dark blue) at the "north" to violet at the "northwest" and then back to red.



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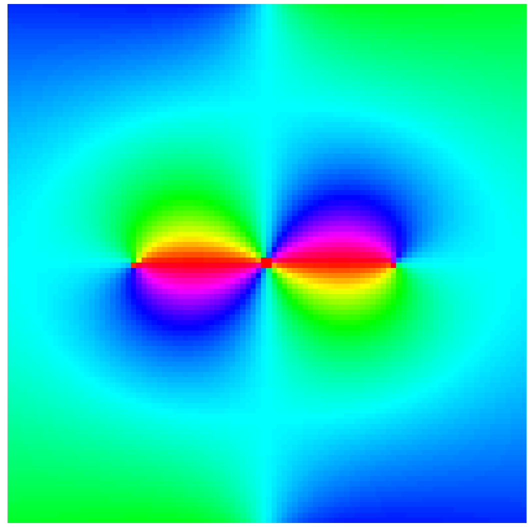
As a little more complicated example is the graph of the function $\frac{z^3+2z^2+z}{z-1}$. In this case we have a single root at $z = 0$, a double root at $z = -1$, and a single pole at $z = 1$. You can tell the double root from the single root because the colors cycle twice around the double root but only once around the single root. This time if we follow the colors around the edge of the graph we end up following them around twice counterclockwise (with a red at both the "south" and the "north" for example).



As a third example, we look at the graph of the function $\frac{z^2-1}{z^4+5z^2}$. In this case we have two single roots at $z = 1$ and $z = -1$. The function has a double pole at $z = 0$. It also has two single poles at $z = i\sqrt{5}$ and at $z = -i\sqrt{5}$, but those are outside the window since $\sqrt{5} > 2$. If we look at the colors around the edge of the graph, we never complete a trip around the rainbow, moving from green to indigo (dark blue) and back again, but never completing the full rainbow. So in this case there are 0 rainbows around the edge.

Now fill in the following table. First use the default window, with both the Real and Imaginary ranges going from –2 to 2. Count how many roots and poles are shown in the graph. Don't count roots or poles that lie outside the graphed region, but do count roots and poles with multiplicity. Also count how many times you cycle through the rainbow as you move around the edge of the graph window. When counting the rainbows around the edge of the graph, count rainbows that move counterclockwise as you go from red to yellow to green to blue to violet as positive and rainbows that go clockwise as negative. Once you have finished that, right-click on the graph (or shift-click if you have a Mac with a one-button mouse), and change the domain to range from –5 to 5 for both the real and imaginary axes.

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	Domain from –2 to 2			Domain from – 5 to 5		
Function	# Roots	# Poles	Rainbows around the edge	# Roots	# Poles	Rainbows around the edge
$\frac{z^2+z}{z-1}$	2	1	1			
$\frac{z^3+2z^2+z}{z-1}$	3	1	2			
$\frac{z^2-1}{z^4+5z^2}$ `	2	2	0			
z^2-5z+4						
z^3-27						
$\frac{1}{z^2-1}$						

1. Look at the values you found in the table above. Can you find a pattern for the rainbows around the edge in terms of the # of roots and the # of poles? This pattern is the principle of the argument

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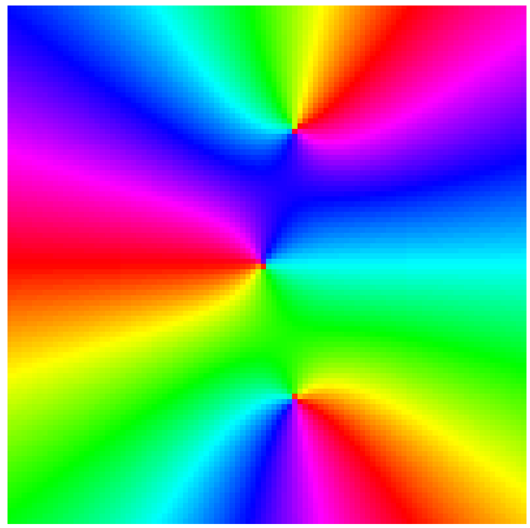
Step 2: Behavior at Infinity

The second piece we will need to justify the fundamental theorem of algebra is a discussion of the behavior of polynomials at infinity. We have already discussed for 2-dimensional real graphs that the behavior of a polynomial at infinity matches the behavior of the leading term. That is still true when you consider the graphs over the complex plane. To see what this means in terms of the "argument," we will zoom out toward infinity and compare the behavior of rainbows around the edge of the graph for various polynomials with the behavior of just the leading terms. In the following examples, we will only have to zoom out (by right-clicking and changing the domain) to have the real and imaginary values range from –10 to 10. With other examples, we might have to zoom out farther, but the general pattern you find will eventually work for any polynomial if you zoom out sufficiently far.



For example, consider the polynomial $z^3 - 2z^2 + 28z + 5$. If we look at its graph on a large domain (say from –10 to 10), we get the following picture. Counting the rainbows, we find there are 3 rainbows moving counterclockwise around the edge of the graph. Now compare this to the graph of just the leading term, z^3 (not pictured). This graph isn't the same in the interior of the graph, but it also has 3 rainbows moving counterclockwise around the edge. Now fill in the following table and look for a pattern.

Polynomial	Rainbows around the edge for a large domain (from -10 to 10)	Leading Term	Rainbows around the edge
$z^2 + 2z - 24$		z^2	
$z^3 - 2z^2 + 28z + 5$	3	z^3	3
$z^4 - 3z^3 + 4z^2 - 80z + 50$			
$z^5 + 3z^3 - 28z$			

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2. How do the number of rainbows for the polynomial compare to the number of rainbows for just the leading term (if you take a sufficiently large window). How many rainbows around the edge of the graph will a polynomial of degree *n* have?

Final step: Putting it all together

In class we discussed that the behavior of a polynomial as x gets large is determined by the leading term. The second part shows this is true for complex polynomials of z as well. We combine this fact with the principle of the argument. From the second pattern, we know that a polynomial of degree n has n rainbows around the edge of the graph. From the first pattern, that means n = # roots – # poles of the polynomial. But a polynomial has no poles (since it has no denominator), so the polynomial must have exactly n roots (including complex roots and counting multiplicity). And this is exactly the Fundamental Theorem of Algebra. In Math 630, you can learn how to prove these patterns must always hold, but just by experimentation I hope you can see these patterns, and then understand why these patterns imply the Fundamental Theorem of Algebra is true.