

Name: \_\_\_\_\_

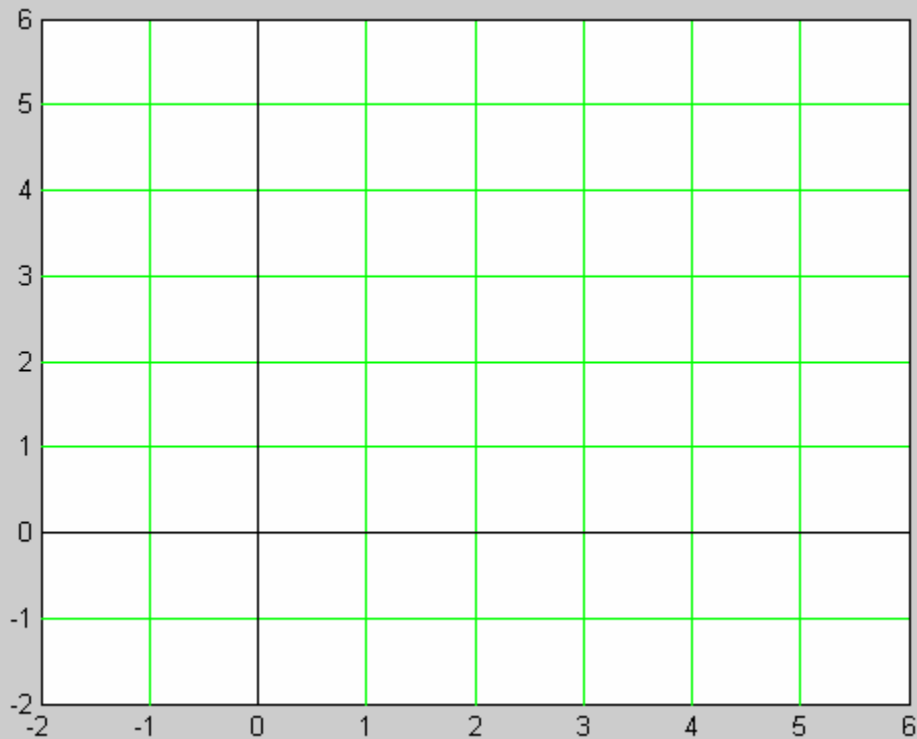
## Complex Graphs Studio

*Euclid alone has looked on Beauty bare. – Edna St. Vincent Millay*

As we have discussed (and danced), the complex numbers are just as real as the real numbers. The difference is that instead of looking at a real number line, we are looking at the complex number plane. Allowing for two-dimensional numbers lets us extend ideas of arithmetic and algebra to many new situations, which is why complex numbers are important in many practical applications. In addition, some features that can be hard to understand if you look at a problem in just one dimension suddenly become much more understandable when you see them in the full higher-dimensional picture. And from an aesthetic standpoint, it also opens up new dimensions of patterns.

1. Let's first be sure you have a sense of where numbers go on the complex plane. Label the following points on the graph:
  - A. 0
  - B.  $3 + i$
  - C.  $4 + 3i$       which is  $(3 + i) + (1 + 2i)$
  - D.  $1 + 2i$

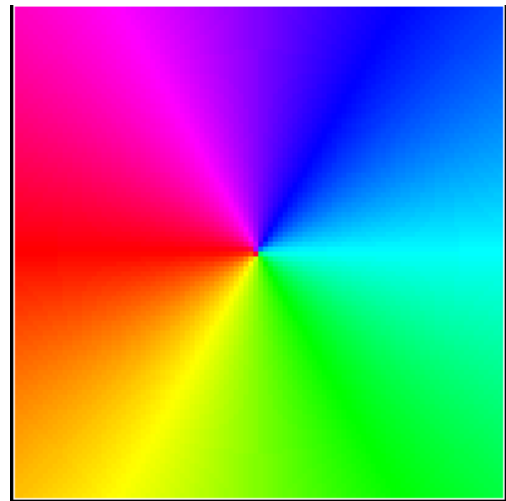
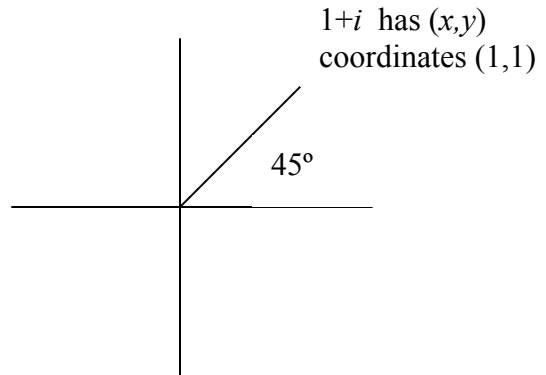
Play connect the dots from A to B to C to D and back to A. *What sort of figure do you draw?*



Name: \_\_\_\_\_

You should have drawn a parallelogram. One way to think of this is that adding numbers in the complex plane is the same as adding numbers on the number line. On the number line, you add two numbers by starting at one number and then moving the distance of the second number. It doesn't matter which you start at, as long as you start at one and move along the other, you end up at the same place. In the same way, if you add two complex numbers, you start at one of them and move along the other one, and you end up at the same place no matter which one you start at and which you move along. It's just that now you are allowed to move in two dimensions instead of just one.

We will now look at some basic graphs of rational functions over the complex plane. Right away this introduces a problem. We normally graph a function by plotting the input variable  $x$  along one number line and matching it with the output variable  $y$  along a perpendicular number line. But if we are looking at functions that take a complex number as input (and then will produce a complex number as output), we need to plot input points on a complex plane matched with output points on a perpendicular complex plane, which requires  $2 + 2 = 4$  dimensions. We will get around this by using color to represent one of the (output) dimensions. We can identify any point in the plane not just by  $x$  and  $y$  coordinates, but also in "polar coordinates" where we note the distance and direction from the origin. So, for example, we could denote the point  $1 + i$  by observing it is  $\sqrt{2}$  units from the origin at a heading of  $45^\circ$  counter-clockwise from the  $x$ -axis. We will plot the input variable on a complex plane. We will then plot the distance of the output variable from the origin (which, being a distance, is a one-dimensional real number) along a perpendicular line. The angle will be designated by the color of the point, with positive real numbers colored light blue, positive imaginary numbers colored dark blue shading to purple, negative real numbers colored red, and negative imaginary numbers colored yellow-green, as illustrated. This will enable us to plot complex functions as a colored three-dimensional plot (which we will project onto a two-dimensional computer screen).



Follow the link to the studio "mathlet." This stuff can't be done with a spreadsheet (unfortunately), so I've programmed an online tool that will let you enter complex functions and see their graphs. Note that the function you enter needs to be written as a function of  $z$ . When dealing with complex numbers, mathematicians often switch to a variable  $z = x + iy$ . You can look at the pictures from the side or from the top. The top view makes it easier to see the colors (corresponding to the angles of the output values).

Name: \_\_\_\_\_

Surprisingly, you lose less information than you might think by just having the colors and not the magnitudes of the output values. If you right-click on the graph, you will bring up a dialog box where you can reset the viewing window, but I'll try to stick to examples where you can just use the default window of  $[-2,2] \times [-2,2]$  for the input variable (with the size of the vertical – which represents the magnitude of the output – automatically adjusting to catch the whole range of the graph).

2. We'll first look at the complex graphs of some polynomials. For each polynomial, compute the zeros of the polynomial. Look at the graph (top view may be best).

- $z$
  
- $z - 1$
  
- $z^2 - 1$
  
- $z^2 + 1$
  
- $z^3 + z^2 - 2$       (hint:  $z = 1$  is one root)

- *What visual clue from the top view shows you that the polynomial has a zero?*

3. Now let us consider double roots. For each polynomial, compute the zeros of the polynomial, and also the degree of each root. Look at the graph (top view may be best).

- $z^2$
  
  
- $z^2 + 2z + 1$

Name: \_\_\_\_\_

- $z^3$
  
  - $(z + 1)^3$
  
  - $z^3 + 1$  (hint:  $z = -1$  is a root)
  
  
  - $z^3 - z^2$
  
  
  - *What visual clue from the top view shows you that the polynomial has a double root, as opposed to a single root? What about triple roots?*
4. Next, we'll move on to rational functions. *Find the zeros and poles of the following functions*, then graph them, viewing them both from the side view and the top view.
- $1/z$
  
  
  
  
  
  
  
  
  
  
  - $(z+1)^3/(z^2 - z)$
  
  
  
  
  
  
  
  
  
  
  - $1/(z + 1)^2$

Name: \_\_\_\_\_

- $(z^3 + 1) / z^2$  (hint:  $z = -1$  is a zero)

- *Based on the side view, why do we call roots of the denominator “poles?”*

- *Based on the top view, what visual clue shows you where poles are? How does this differ from the graph of a zero? You may find it useful to look at the graphs of  $z$  and  $1/z$  again in answering this last part.*

5. Once you have the ideas about zeros and poles for rational functions, can you *identify the two functions graphed at the top of the next page* just from their top views? Note that you will probably want to look at a color version of this handout rather than a black & white copy (either by looking at this page online, or printing it off on a color printer). Note that your monitor almost surely has more color depth than your printer, which means the colors may look slightly different on screen than on the page.

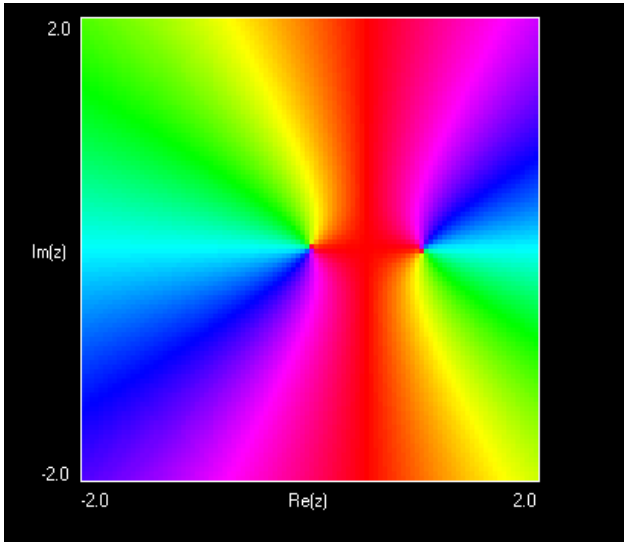
A. In Function A, the points of interest are at 0 and 1. Identify which are zeros and which are poles from the picture, then you should be able to identify the function. Enter the function into the “mathlet” to check you are right.

B. In Function B, the points of interest are 1,  $-1$ ,  $i$ , and  $-i$ . Identify which are zeros and which are poles and you should be able to identify the function. Enter the function into the mathlet to check you are right.

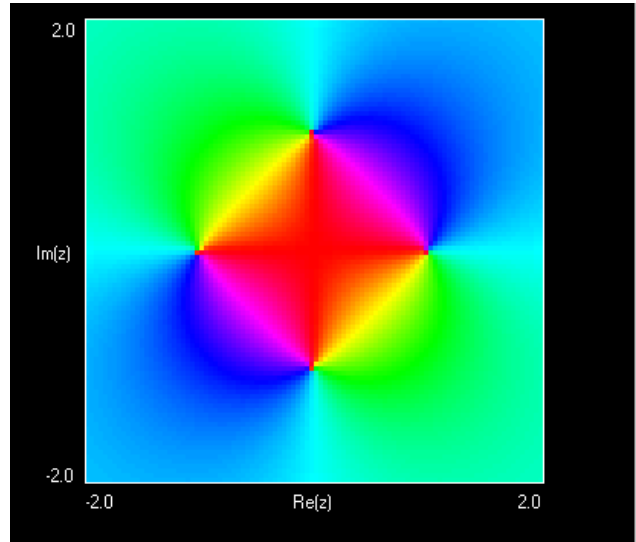
Function A is  $f(z) =$

Function B is  $f(z) =$

Name: \_\_\_\_\_



Function A



Function B

Bonus

6. The final assignment is to *draw a pretty picture*. Turn in a function that makes an aesthetically pleasing image when you graph it. People usually find top views prettier than side views, but you may indicate if I should look at the function using the side view instead. You should also indicate if the window needs to be changed to see everything of interest. Of course, you need to mathematically analyze your function. **Indicate how many zeros and poles there are in your picture, including the degree of each zero and pole.** Note that you can enter functions in factored form (e.g. it is okay to enter something like  $f(z) = (z+1)(z-2)^2 / (z^3 + 5)$  into the mathlet) which should make it easier for you to create the picture you want.

Name: \_\_\_\_\_

I said the first day that while we would focus on algebraic skills and using algebra to interpret and predict real-world situations, you should also expect to learn a little bit about the culture of mathematics. Mathematics is *not* just learning lots of different algorithms for solving different problems. It provides a way of looking at the world that is both useful and beautiful. When you see how mathematical ideas fit together, they snap into place with the same sort of pleasure that you can get by solving a crossword or a Sudoku puzzle, though much more intense. I hope this studio has not just given you a sense of complex numbers and rational functions, but also a sense of the beauty of mathematics. And these patterns aren't just aesthetically pleasing. The patterns formed by zeros and poles that we have discussed in this studio lead to a significant result called the "Principle of the argument." This theorem has practical applications in stability theory, as well as many interesting uses in theoretical mathematics. In particular, the fundamental theorem of algebra, that every polynomial has exactly as many roots as its degree (assuming you count things correctly and include complex roots) is a consequence of these sorts of ideas. For (at least some of the) details of this, see the extra credit.