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TRANSFORMATIONS: NEW FUNCTIONS FROM OLD ONES

The Problem: Given some function $f(x)$, we build a new function $g(x)$ out of $f(x)$, using certain *Transformations*. ***Want:*** *graph of* $g(x)$ *in relation to the graph of* $f(x)$.

The table below summarizes the types of transformations and their effect on the graphs[1]

$g(x)$	Graph of $g(x)$ from graph of $f(x)$
$f(x)+k$	*Shifted vertically by* k *units*
$f(x+h)$	*Shifted horizontally by* $-h$ *units* **(!)**
$af(x)$	*Dilated vertically by a factor of* a
$f(ax)$	*Dilated horizontally by a factor of* $1/a$ **(!)**
$-f(x)$	*Reflected with respect to the* x*-axis*
$f(-x)$	*Reflected with respect to the* y*-axis*

Tips: The list below explains the meaning of several geometric transformations listed in the second column in the table above:

- a vertical shift by a *positive* number means ***upward***,
- a vertical shift by a *negative* number means ***downward***,
- a horizontal shift by a *positive* number means ***to the right***,
- a horizontal shift by a *negative* number means ***to the left***,
- a dilation by a *factor greater than* 1 means ***stretching***,
- a dilation by a *factor less than* 1 means ***shrinking***.

In the four examples below, the original function is $f(x) = x^2$, whose graph will be shown in blue in all figures. In several instances some reference points on the graph of $f(x)$ will also be identified. The new function $g(x)$ (and the transformed reference points) will be shown in red.

[1] Be aware of the transformation marked with a **(!)**.

EXAMPLE 1: The graph of $g(x) = (x+2)^2$ is obtained from the graph of $f(x) = x^2$ by means of a horizontal shift by -2. This amounts to a *left translation*. Under this transformation, the point $(0,0)$, on the graph of $f(x)$, corresponds to the point $(-2,0)$, on the graph of $g(x)$. This is illustrated in *Figure 1* on the right.

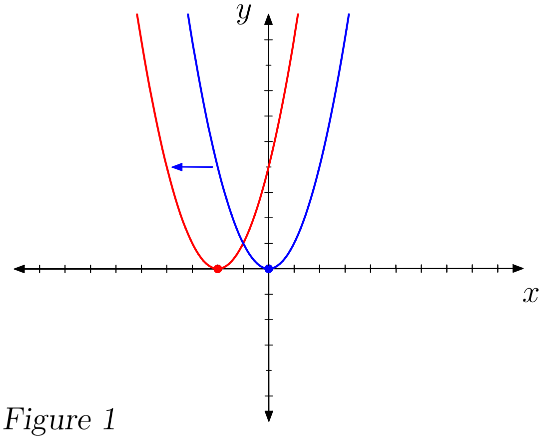

Figure 1

EXAMPLE 2: Consider the function $g(x) = -x^2$. The graph of $g(x)$ is obtained from the graph of $f(x) = x^2$ by means of an x-axis reflection. Under this transformation, the point $(2,4)$, on the graph of $f(x)$, corresponds to the point $(2,-4)$, on the graph of $g(x)$. This is illustrated in *Figure 2* on the right.

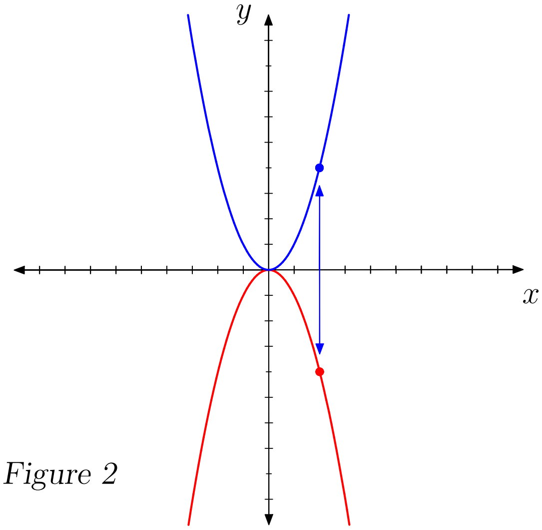

Figure 2

EXAMPLE 3: Consider the function $g(x) = -x^2 + 4$. To obtain the graph of $g(x)$, we first consider the function $h(x) = -x^2$, whose graph was already discussed in Example 2. The graph of $g(x)$ then is obtained from the graph of $h(x)$ by means of vertical shift by 4 units. This amounts to an *upward translation*.

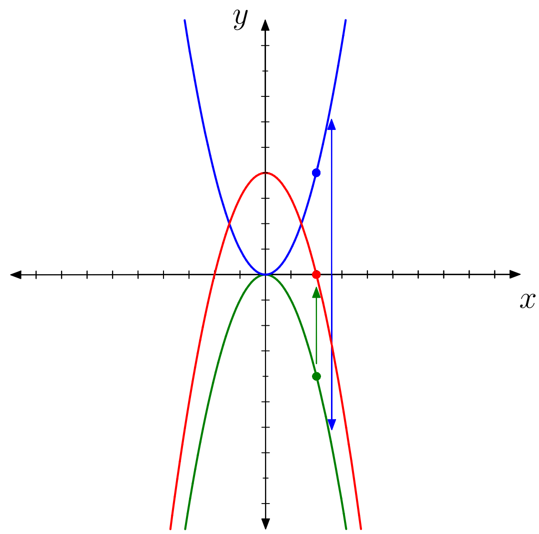

Figure 3

Under this two-step transformation (x-axis reflection, followed by upward shift by 4), the point $(2, 4)$, on the graph of $f(x)$, first corresponds to the point $(2, -4)$ on the graph of $h(x)$, then corresponds to the point $(2, 0)$, on the graph of $g(x)$. *Figure 3* shows the graphs of all three functions: $f(x)$, $h(x)$, and $g(x)$.

EXAMPLE 4: Consider the function $g(x) = (3x)^2$. The graph of $g(x)$ then is obtained from the graph of $f(x)$ by means of horizontal dilation by a factor of 1/3. Since the dilation factor is less than 1, this amounts to a *shrinking of the graph in the horizontal direction*. Under this transformation, the point $(1, 1)$, on the graph of $f(x)$, corresponds to the point $(\frac{1}{3}, 1)$, on the graph of $g(x)$. This is illustrated in *Figure 4* on the right.

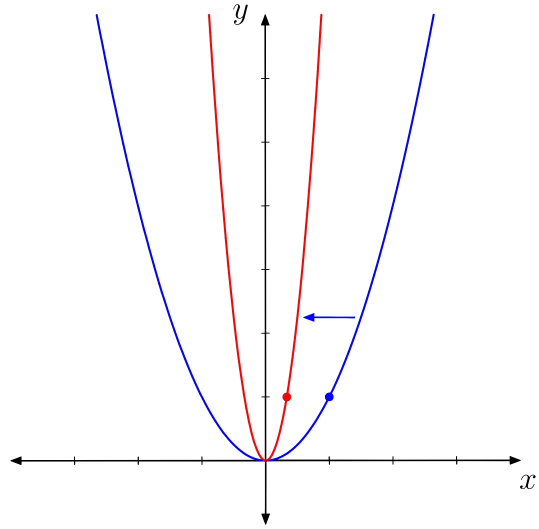

Figure 4

REMARK: The function considered in Example 4 can also be written as $g(x) = 9x^2$, which means that the graph of $g(x)$ can also be obtained from the graph of $f(x)$ by a vertical dilation by a factor of 9. This will amount to a *stretching of the graph in the vertical direction*.