PHASE PORTRAITS AND CONFIGURATIONS FOR HAMILTONIANS ON A SPHERE

A.K. WHEELER

1. Introduction

Given a symplectic surface, is there a Hamiltonian differential equation, H, with any predetermined equilibrium set? The physical motivation for these problems comes from classical Hamiltonian mechanics. Hamilton's equations are differential equations which relate the total energy of a system to its momentum and position. Momentum is plotted against position to give the *phase space* of the system; level curves in the phase space are solutions to the differential equations, which coincide with the orbits of the Hamiltonian vector field. In this paper we only deal with two types of singularities. *Elliptic* singularities are given by equations of the form $H = p^2 + q^2$ and have index $+1$. *Hyperbolic* singularities are given by equations of the form $H = pq$ and have index -1. Symplectic manifolds must satisfy the *Poincare-Hopf Theorem.*

Theorem 1.1. *Poincaré-Hopf. For a compact smooth manifold S with vector field v having isolated singularities, the following formula holds:*

$$\sum_i index_v\left(x_i\right) = \chi\left(S\right)$$

where the sum of the indices is over all the isolated singularities of v and $\chi\left(S\right)$ is the Euler characteristic of S.

This small paper deals with a specific case of smooth compact surfaces of genus 0 (i.e., spheres), though a later section will generalize the results to tori and surfaces of higher genus.

Definition 1.2. *A sphere is assigned a configuration determined by an ordered pair, $\{p, n\}$, where p is the number of elliptic singularities present in the Hamiltonian, H, and n is number of hyperbolic singularites.*

To begin, we consider H as the *height function* for the sphere. Later on, H will be represented by more general *phase portraits.* Hamiltonians H and H' may have the same configuration.

2. Round Spheres and Perturbed Spheres

Given the configuration $\{2, 0\}$ and the phase portrait as the height function for the (round) sphere, elliptic singularities are at the North and South poles. By the Poincaré-Hopf Theorem, any sphere with the height function must have at least 2 elliptic singularites. Notice $p - n = 2$, the Euler characteristic of the sphere.

Date: June 9, 2008.

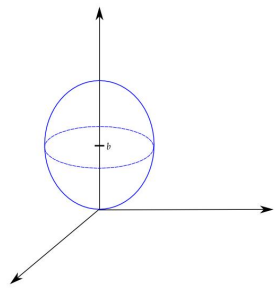

FIGURE 1. The height function for this sphere is the distance up from the origin. At the center of the sphere, $H = b$.

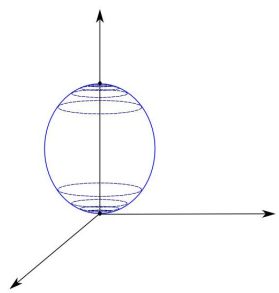

FIGURE 2. The dotted circles are the level curves on the sphere, defined by the height function. At the North and South Poles are elliptic singularities.

An elliptic singularity on a level curve is denoted c. Circles on the level curves are denoted C. An increase in n implies an increase in p, so for example, $n = 1$ implies $p = 3$. In this case, using the height function again, the phase portrait is on the sphere with a perturbation. The local minimum is the new hyperbolic singularity, denoted I_1. When there are m hyperbolic singularities on the same connected component of a level curve, they are denoted I_m.

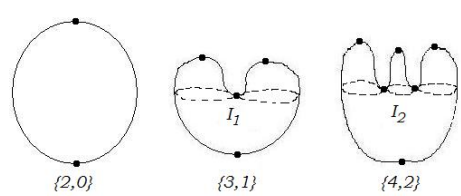

FIGURE 3. Configurations $\{2, 0\}$, $\{3, 1\}$, and $\{4, 2\}$, showing zero, one, and two perturbations, respectively.

3. CONFIGURATIONS

Using the height function for Hamiltonian H on the sphere, we saw any certain configuration $\{p, n\}$ provides a 1st Combinatorial Invariant, in other words, for a given $\{p, n\}$ there is an equivalence class of pairs (S^2, H), where S^2 is a 2-sphere

and H is a Hamiltonian with p elliptic singularities and n hyperbolic singularities, such that

$$(S^2, H) \sim (\tilde{S^2}, \tilde{H}) \text{ if } p = \tilde{p} \text{ and } n = \tilde{n}.$$

A level curve consisting of unions of C, c, and I_m is called a *fiber*. Unions of all fibers make up the *fibration* of a surface. By looking at the fibration $\mathcal{F}_H$ we will introduce a 2nd Topological Invariant. Suppose we have two surfaces $(\Sigma, \mathcal{F}_H)$ and $(\Sigma', \mathcal{F}_{H'})$ with the same configuration, $\{p, n\}$, and the following diagram commutes:

$$\begin{array}{ccc} (\Sigma, \mathcal{F}_H) & \xrightarrow{\Phi} & (\Sigma', \mathcal{F}_{H'}) \\ \downarrow H & & \downarrow H' \\ I & \xrightarrow{\phi} & I' \end{array}$$

where I and I' are intervals on the real line. $H' \circ \Phi = \phi \circ H$ so that Φ sends fibers of H to fibers of H', where Φ and ϕ are homeomorphisms and H and H' are Hamiltonians.

Notation. Σ_a denotes the fiber on Σ which gets mapped by H to the point a.

Besides the height function H, there is another way to realize certain configurations of singularities. From this point on, we take maps onto a circle. First however, we must be able to represent spheres by trees. For a given fibration, map each circle to a point and each hyperbolic singularity to an inner vertex. An inner vertex is a vertex connected to at least three edges. Outer vertices are vertices connected to exactly one edge. By induction, any configuration $\{p, n\}$ can be represented this way. The configuration $\{2, 0\}$ is simply an edge, since there are no hyperbolic singularities. The base step, $\{3, 1\}$ is a vertex connected to three edges.

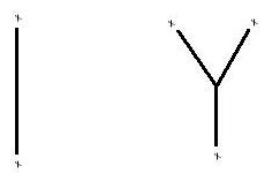

FIGURE 4. Tree representations for $\{2, 0\}$ and $\{3, 1\}$.

Now suppose a tree representation exists for $\{p, n\}$. For $p+1$, we need one more edge to obtain one more outer vertex, and a new inner vertex to depict the increase in n. There are two possibilities for this part of the construction.

(1) Attach an edge to an interior part of another edge. This action adds one I_1 vertex and one outer vertex, so is permissible. However, the new edge must not intersect any other existing ones, because if it does, the graph will no longer be a tree. Cycles in the graph correspond to surfaces with a higher genus, considered in a later section of this paper.
(2) Attach an edge to an inner vertex. The inner vertex I_m becomes I_{m+1} and another c is added.

Proposition 3.1. *Any type of singular fiber on a sphere can be realized with a map to a circle.*

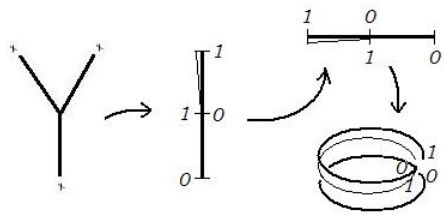

FIGURE 5. A $\{3,1\}$ sphere mapped to a tree, then a line segment, then a circle. In this case, the map is not injective, but in general the map need not be surjective at all.

Proof. Using the technique above, first map the configuration to a tree. The tree gives a branched cover of the segment $[0,1]$. Identifying the points 0 and 1 gives a branched cover of S^1. In order to realize different fibrations, two modifications can be made to the map. First, the tree can be mapped to $[0,1]$ α times, where α is any positive number, forming an α-cover. Second, the lengths of the edges of the tree can be changed so that the branches only cover certain parts of the circle.

□

4. Fibrations and Fibers

This section looks at the 2nd Topological Invariant in more detail, the possible fibrations given a configuration. There are three types of basic fibers to consider. All realizations will be the circle map described in the last section. The three types of fibers c, C, and I_m can appear as unions together in one fiber. In fact, provided the Poincaré-Hopf Theorem is satisfied, components of type c and type I_m can appear in a fiber together in any combination. Finally, arbitrarily large numbers of components of type C can appear in any fiber, since this corresponds to wrapping an edge around the circle arbitrarily many times. This gives rise to having multiple copies of components of type C, i.e., multiple fibers. Some questions to consider:

(1) Can we realize all types of fibers without introducing multiple fibers?
(2) What is the smallest number of components of type C needed to realize a singular fiber of a given type?

Example. For $\{2,0\}$ only three basic fibers are interesting to consider, particularly, fibers containing exactly 0,1, or 2 components of type c, since no hyperbolic singularities are available and components of type C can always be added. For a fiber with zero elliptic singularities at least one component of type C must be present, since fibers are always non-empty. A fiber with exactly one elliptic singularity can appear, provided the tree is only mapped to a portion of the circle. If the tree is mapped to the circle more than once, a component of type C must also appear. Finally, to map both ellptic singularities to the same point the minimum number of components of type C is zero, since every other point of the tree can cover exactly one point of the circle.

The following table presents the minima for configurations with small numbers of singularities.

$\{p,\ n\}$	C	c	I_1	I_2	I_3
$\{2,\ 0\}$	1	0			
	1	1			
	1	2			
$\{3,\ 1\}$	2	0	0		
	2	1	0		
	2	2	0		
	2	3	0		
	2	0	1		
	2	1	1		
	3	2	1		
	3	3	1		
$\{4,\ 2\}$	2	0	0	0	
	2	1	0	0	
	2	2	0	0	
	2	3	0	0	
	3	4	0	0	
	2	0	1	0	
	2	1	1	0	
	3	2	1	0	
	4	3	1	0	
	4	4	1	0	
	5	4	2	0	
	3	0	0	1	
	3	1	0	1	
	3	2	0	1	
	3	3	0	1	
	4	4	0	1	

5. Higher Genera

In the previous section it was mentioned that outer vertices of trees carry a plus sign to denote elliptic singularities. Suppose in a tree, exactly two of these vertices appear without a plus sign. By convention, this means we are identifying the two extrema.

Example. Consider the tree for a $\{3, 1\}$ sphere, and remove the plus signs from the northernmost branches. When these pieces are identified the result is topologically equivalent to a torus, or a sphere with a "handle". Compare this to the notion of the perturbed sphere where the phase portrait is the height function.

This is a $\{2, 2\}$ torus. Notice $2 - 2 = 0$, the Euler characteristic for the torus. The top part of the loop simultaneously represents a maximum and a minimum. The bottom part of the loop has a stem, so the hyperbolic singularity is at the

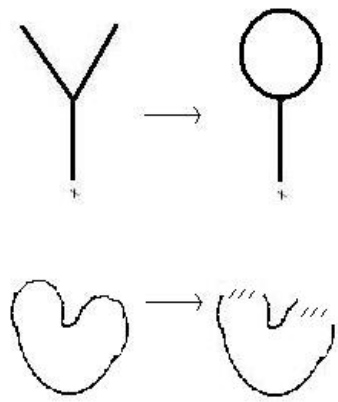

FIGURE 6. Removing plus signs can identify sides and create surfaces of higher genus. The hatch marks indicate pieces which are identified.

vertex, as usual, and the remaining elliptic singularity is at the bottom, denoted by a plus sign.

To create surfaces of higher genus, more than two outer vertices must be identified. This occurs in pairs, and pairs of outer vertices are to be identified with upper case letters. If a lone outer vertex does not have another with which to identify, the surface is "open", until it can be glued somewhere else.

All that was discussed in the previous section about mapping trees to circles is generalized here. The only difference is that now trees are allowed to have cycles, i.e., they become graphs.

6. Further Work

(1) Suppose ω is the volume of the given surface Σ with phase portrait h and configuration $\{p, n\}$. Under what conditions is volume preserved?
(2) What information can we learn from looking at other phase portraits? For example, what if we represent surfaces as polygons with appropriate sides identified?

7. Acknowledgements

Department of Mathematics, Kansas State University
E-mail address: leyjfk6@ksu.edu