Math 994
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"Dynamical systems method for solving operator equations"
The material will be taken from the book:

A.G. Ramm, Dynamical systems method for solving operator equations, Elsevier, Amsterdam, 2006.

Consider an operator equation F(u)=0 in a Hilbert space H and assume that this equation is solvable, possibly nonuniquely. Let us call the problem of solving this equation ill-posed if the operator $F'(u)$ is not boundedly invertible, and well-posed otherwise. A general method, Dynamical Systems Method (DSM), for solving linear and nonlinear ill-posed problems in H is presented. This method consists of the construction of a dynamical system, that is, a Cauchy problem,

$$\dot{u} = \Phi(t, u), \quad u(0) = u_0,$$

which has the following properties:

1) it has a global solution, i.e., its solution is defined on $[0, \infty)$,

2) this solution tends to a limit as time tends to infinity, i.e., there exists the limit $\lim_{t\to\infty} u(t) = u(\infty)$,

and

3) this limit solves the original equation, i.e., $F(u(\infty)) = 0$.

The choices of $\Phi(t, u)$ are proposed and the DSM is justified for

a) arbitrary solvable linear equations of the form $Au = f$ with densely defined closed linear operator A,

b) for well-posed nonlinear equations with twice Fréchet differentiable operator F,

c) for ill-posed nonlinear equations with monotone operators,

d) for ill-posed nonlinear equations with non-monotone operators such that $F'(y) \neq 0$, where $F(y) = 0$,

e) for operators such that $A := F'(u)$ satisfies the spectral assumption:

$$||(A + sI)^{-1}|| \leq c/s,$$

where $c > 0$ is a constant, and $s \in (0, s_0), s_0 > 0$ is a fixed number, arbitrarily small, c does not depend on s and u,

f) for some monotone operators which are not Fréchet differentiable, but only hemi-continuous and defined on all of H,

g) for some unbounded, closed, densely defined F, in particular, for some semilinear elliptic problems,

and

h) for some operator equations in Banach spaces.

k) In Newton-type schemes the main difficulty is to invert the derivative of the operator. A novel scheme, based on the DSM, allows one to avoid this inversion.

A global convergence theorem is obtained for the regularized continuous analog of Newton's method for monotone operators. Global convergence means that convergence is established for an arbitrary initial approximation, not necessarily the one which is sufficiently close to the solution.

A general approach to constructing convergent iterative schemes for solving well-posed nonlinear operator equations is described and convergence theorems are obtained for such schemes.

Stopping rules for stable solution of ill-posed problems with noisy data are given.

References

[1] A.G.Ramm, *Dynamical systems method for solving operator equations,* Communic. in Nonlinear Sci. and Numer. Simulation, 9, N2, (2004), 383-402.

[2] A.G.Ramm, **Inverse Problems**, Springer, New York, 2005.

[3] A.G.Ramm, **Dynamical systems method for solving operator equations**, Elsevier, Amsterdam, 2006.