KANSAS STATE UNIVERSITY

The Tenth

S. Thomas Parker Mathematical Competition
April 12, 2008

Instructions: Put your name on all papers you use and turn them all in. Try to solve as many problems as you can. For any problem you try, give as complete an answer as you can. Include a clearly written explanation of how you found your answer and why it is true. You may use drawings or calculations to help you for your justification, but your explanation should be convincing. You may not use calculators.

1. Sketch the graph of the equation

$$x^y = y^x \qquad x > 0,\ y > 0.$$

   Find the slope of the tangent line to the graph at the points $(2, 4), (1, 1)$. What happens at the point (e, e)?

2. Find the maximum possible surface area (side, top, and bottom) of a right circular cylinder which is inscribed in a sphere of radius R.

3. Prove that $e^x \leq x + e^{x^2}$ for all real numbers x. (Recall that $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$.)

4. For $\frac{\pi}{4} \leq \theta \leq \frac{3\pi}{4}$, show that

$$\sqrt{1 + \sin 2\theta} + \sqrt{1 - \sin 2\theta} = 2 \sin \theta.$$

   What does the left side simplify to on the interval $\frac{-\pi}{4} \leq \theta \leq \frac{\pi}{4}$? Use this to draw the graph of the function

$$f(\theta) = \sqrt{1 + \sin 2\theta} + \sqrt{1 - \sin 2\theta}$$

   over the interval $[-2\pi, 2\pi]$.