KANSAS STATE UNIVERSITY

The Nineth Annual

S. Thomas Parker Mathematical Competition
March 31, 2007

Instructions: Put your name on all papers you use and turn them all in. Try to solve as many problems as you can. For any problem you try, give as complete an answer as you can. Include a clearly written explanation of how you found your answer and why it is true. You may use drawings or calculations to help you for your justification, but your explanation should be convincing.

1. a) A K-State engineering student is designing a bridge over the Kansas river with the main frame of the bridge consisting of two parabolas. In order to succeed the parabola $y = x^2$ is translated vertically to a parabola $y = x^2 + k$ in such a manner that the translated parabola has exactly two points of intersection with the parabola $y = -2x^2$ and that at each point of intersection the parabolas meet at a right angle, that is, the tangent lines are perpendicular. Find all values of k for which this happens.

   b) Suppose now that the parabola $y = x^2$ is translated to a parabola $y = (x - h)^2 + k$. Show that the only way the translated parabola can meet $y = -2x^2$ in right angles is if $h = 0$, that is, the two parabolas share the same axis.

2. Recall that Pascal's Triangle is formed by the coefficients of x^k in the expansion of $(1+x)^n$ with $n = 0, 1, 2, 3, \ldots$ and $0 \le k \le n$. The first few rows of the triangle are given below,

$$\begin{array}{lccccccccc}
n=0 & & & & & 1 & & & & \\
n=1 & & & & 1 & & 1 & & & \\
n=2 & & & 1 & & 2 & & 1 & & \\
n=3 & & 1 & & 3 & & 3 & & 1 & \\
n=4 & 1 & & 4 & & 6 & & 4 & & 1
\end{array}
\qquad\qquad
\begin{array}{ccccc}
 & & & & \\
 & 1 & & 2 & \\
1 & & 3 & & 3 \\
 & 4 & & 6 & \\
 & & & &
\end{array}$$

   Each internal entry (not on the edge) of the triangle has six entries that are adjacent to it and form six vertices of a hexagon as shown in the example above on the right for the entry 3 in the row $n = 3$. Show that the product of the numbers at three nonadjacent vertices of the hexagon equals the product of the other three numbers on the hexagon. In the example above, $1 \cdot 3 \cdot 4 = 1 \cdot 2 \cdot 6$.

3. Find all nonnegative differentiable functions $f(t)$ defined for $t \ge 0$ such that for any positive real number x the arc length of the curve $y = f(t)$ over $[0, x]$ equals the area under the curve $y = f(t)$ and above the t-axis over the interval $[0, x]$.

4. Five spherical cannon balls of different diameters are placed into a right circular cone so that each ball touches the cone at a circle around the side of the cone and touches the adjacent balls as well. If the smallest ball has diameter 8 and the largest ball has diameter 18, find the diameters of the other three balls.