KANSAS STATE UNIVERSITY

The Eighth Annual

S. Thomas Parker Mathematical Competition
April 1, 2006

Instructions: Put your name on all papers you use and turn them all in. Try to solve as many problems as you can. For any problem you try, give as complete an answer as you can. Include a clearly written explanation of how you found your answer and why it is true. You may use drawings or calculations to help you for your justification, but your explanation should be convincing.

1. An air-fire station is located a miles due North of a point P on a river that runs due East. A popular campground is located at a point c miles due North of a point Q on the river. Point Q is b miles East of the point P. To respond to a fire at the campground, a helicopter has to fly to the river to pick up water and then fly to the fire with the water. The helicopter flies at 100 miles per hour without water and only 60 miles per hour with the water. The firemen have determined that the fastest way to get to the fire is to fly straight to a point on the river $b/2$ miles East of the point P, and then fly straight to the campground. Determine a very simple relationship between a, b and c.

2. Let $a_1, a_2, \ldots$ be a sequence of positive integers. For any positive integer n, define a function
$$f_n(x) = \frac{a_1 + \cdots + a_{n-1} + x}{n} \cdot \frac{1}{\sqrt[n]{a_1 a_2 \cdots a_{n-1} x}}, \qquad (x > 0).$$
(a) Find the critical point of $f_n(x)$ and prove that $f_n(x)$ attains a minimum value at the critical point.

   (b) Prove by induction (or any other means) that for any positive integer n

   (i) $f_n(x) \geq 1$ for all $x > 0$ and

   (ii) $\frac{a_1 + \cdots a_{n-1} + a_n}{n} \geq \sqrt[n]{a_1 a_2 \cdots a_{n-1} a_n}$.

3. Consider an acute triangle ΔABC. Let A' be the point on the side $\overline{BC}$ such that the line segment $\overline{AA'}$ is perpendicular to the side $\overline{BC}$. Similarly let B' and C' be the points on sides $\overline{AC}$ and $\overline{AB}$ respectively such that $\overline{BB'}$ is perpendicular to $\overline{AC}$, and $\overline{CC'}$ is perpendicular to $\overline{AB}$.

   a) Prove that each of the triangles $\Delta A'BC'$ $\Delta B'CA'$, and $\Delta C'AB'$ are similar to the original triangle ΔABC. (You may assume that the segments $\overline{AA'}$, $\overline{BB'}$, $\overline{CC'}$ meet at a common point.)

   b) Draw an obtuse triangle and state and prove a similar statement.

4. For each pair of non-negative integers $n \geq k \geq 0$, define
$$\binom{n}{k} = \frac{n \cdot (n-1) \cdot (n-2) \cdots (n-k+1)}{k(k-1) \cdots 2 \cdot 1}.$$
For any positive integer n, compute $\sum_{i=0}^{n} (-1)^i \binom{n}{i}^2$. (Hint: $1 - x^2 = (1+x)(1-x)$.)