KANSAS STATE UNIVERSITY

the Seventh

S. Thomas Parker Mathematical Competition
April 16, 2005

Instructions: Put your name on all papers you use and turn them all in. Try to solve as many problems as you can. For any problem you try, give as complete an answer as you can. Include a clearly written explanation of how you found your answer and why it is true. You may use drawings or calculations to help you for your justification, but your explanation should be convincing.

1. Let C be the parameterized plane curve $x = \sin(t)$, $y(t) = \cos(t) + \ln(\tan(\frac{t}{2}))$, $0 < t < \pi$. For each t, let $D(t)$ be the distance between the point $(x(t), y(t))$ and the intersection of the y-axis with the tangent line to C at the point $(x(t), y(t))$. Derive a simple formula for $D(t)$.

2. Obtain explicit formulas for

$$\int_0^{\frac{\pi}{2}} \sin^{2m}(x)dx \quad \text{and} \quad \int_0^{\frac{\pi}{2}} \sin^{2m+1}(x)dx$$

and use them to find a real number $P(m)$ for each positive integer m such that

$$\frac{\pi}{2} = \frac{2 \cdot 2}{1 \cdot 3} \cdot \frac{4 \cdot 4}{3 \cdot 5} \cdot \frac{6 \cdot 6}{5 \cdot 7} \cdots \frac{2m \cdot 2m}{(2m-1) \cdot (2m+1)} P(m).$$

Prove that $\lim_{m \to \infty} P(m) = 1$.

3. John Conway visited K-State ten years ago and gave a sequence of numbers

$$1, 11, 21, 1211, 111221, 312211, 13112221, 1113213211, \ldots$$

in which each number is obtained by reading the previous one. For example, to produce the next term, 1211 is read as one one, one two, two one(s). Determine (with proof) all possible digits that can appear in these numbers. For example, just the digits 1,2,3 appear in the numbers listed above.

4 Let ΔABC be any triangle in the plane. Based on each edge of ΔABC, construct an equilateral triangle exterior to ΔABC. Show that the centers of the three equilateral triangles form the vertices of an equilateral triangle.