KANSAS STATE UNIVERSITY
**Department of Mathematics**

# S. Thomas Parker Mathematical Competition
# April 17, 2004

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**Instructions:** Put your name on all papers you use and turn them all in. Try to solve as many problems as you can. For any problem you try, give as complete an answer as you can. Include a clearly written explanation of how you found your answer and why it is true. You may use drawings or calculations to help you for your justification, but your explanation should be convincing.

1. A cylinder is inscribed in a given sphere of unit radius. Find the dimensions of the cylinder so that the surface area (including the top and bottom) of the cylinder is maximal. **Solution:** Let $r$ and $h$ be the radius and height of the cylinder. Then $r^2 + (\frac{h}{2})^2 = 1$, or $h = 2\sqrt{1-r^2}$. The surface area is

$$A(r) = 2\pi rh + 2\pi r^2 = 4\pi r\sqrt{1-r^2} + 2\pi r^2$$

with $0 \le r \le 1$. To maximize $A(r)$ on $[0,1]$ we take

$$\begin{aligned} A'(r) &= 4\pi(\sqrt{1-r^2} - \frac{r^2}{\sqrt{1-r^2}} + r) \\ &= 4\pi(1-2r^2 + r\sqrt{1-r^2})/\sqrt{1-r^2}. \end{aligned}$$

Solving the equation $A'(r) = 0$,

$$\begin{gathered} 1 - 2r^2 = -r\sqrt{1-r^2} \\ 4r^4 - 4r^2 + 1 = r^2(1-r^2) = r^2 - r^4 \\ 5r^4 - 5r^5 + 1 = 0 \\ r = \tfrac{5\pm\sqrt{5}}{10}. \end{gathered}$$

When $r^2 = \frac{5-\sqrt{5}}{10} < \frac{1}{2}$, then $1-2r^2 \neq -r\sqrt{1-r^2}$ and $r = \sqrt{\frac{5-\sqrt{5}}{10}}$ is not a solution for $A'(r) = 0$. Hence $r = r = \sqrt{\frac{5+\sqrt{5}}{10}}$ is the only critical point in $[0,1]$. We know that $A(0) = 0$ and $A(1) = 2\pi$.

$$\begin{aligned} A(\sqrt{\frac{5+\sqrt{5}}{10}}) &= 4\pi\sqrt{\frac{5+\sqrt{5}}{10}}\sqrt{1 - \frac{5+\sqrt{5}}{10}} + 2\pi\frac{5+\sqrt{5}}{10} \\ &= 4\pi\sqrt{\frac{5+\sqrt{5}}{10}\cdot\frac{5-\sqrt{5}}{10}} + 2\pi\frac{5+\sqrt{5}}{10} \\ &= 4\pi\sqrt{\frac{2}{10}} + 2\pi\frac{5+\sqrt{5}}{10} = 2\pi(2\sqrt{\frac{2}{10}} + \sqrt{\frac{5-\sqrt{5}}{10}}) \ge 2\pi \end{aligned}$$

since $(2\sqrt{2} + \sqrt{5-\sqrt{5}})^2 > (2\sqrt{2})^2 + (\sqrt{5-\sqrt{5}})^2 = 8 + 5 - \sqrt{5} > 10$.

Hence at $r = \sqrt{\frac{5+\sqrt{5}}{10}}$ takes the maximal value. In this case the height of the cylinder is $h = 2\sqrt{\frac{5-\sqrt{5}}{10}}$.

2. Compute the following limit

$$\lim_{n\to\infty}\left(\frac{1}{n}\prod_{i=1}^{n}(n+i)^{\frac{1}{n}}\right).$$

**Solution:** Since $f(x) = e^x$ are continues functions, we have

$$\lim_{n\to\infty}\frac{1}{n}\prod_{i=1}^{n}(n+i)^{\frac{1}{n}} = e^{\lim_{n\to\infty}\ln(\frac{1}{n}\prod_{i=1}^{n}(n+i)^{\frac{1}{n}})}.$$

$$\ln(\frac{1}{n}\prod_{i=1}^{n}(n+i)^{\frac{1}{n}}) = \ln(\prod_{i=1}^{n}(\frac{n+i}{n})^{\frac{1}{n}}) = \sum_{i=1}^{n}\frac{1}{n}\ln(1+\frac{i}{n}).$$

Since the function $\ln(1+x)$ is continuous on $[0,1]$,

$$\lim_{n\to\infty}\sum_{i=1}^{n}\frac{1}{n}\ln(1+\frac{i}{n}) = \int_0^1 \ln(1+x)dx = \int_1^2 \ln(x)dx = x\ln(x)|_1^2 - \int_1^2 dx = 2\ln(2)-1 = \ln(4/e).$$

Thus

$$\lim_{n\to\infty}\frac{1}{n}\prod_{i=1}^{n}(n+i)^{\frac{1}{n}} = e^{\ln(4/e)} = \frac{4}{e}.$$

3. For a triangle $\Delta ABC$ and a straight line $L$ (in the same plane as $\Delta ABC$) intersecting the interior of the triangle $\Delta ABC$, let $\Delta A'B'C'$ be the reflection of the triangle $\Delta ABC$ with respect to the reflection axis $L$ in the plane. Show that one can always choose the line $L$ so that the area of the intersection of $\Delta ABC$ and $\Delta A'B'C'$ is at least $2/3$ of the area of $\Delta ABC$.

There are two proofs by the contestants.

**Proof 1.** Let $a, b, c$ be the lengths of the sides apposite to vertices $A, B, C$ respectively. We can assume that $a \geq b \geq c$. Let $L$ be the line bisecting the angle $\angle C$ intersecting $AB$ at $D$. The reflection image $A'$ of $A$ lies on $BC$. Then the intersection of $\Delta ABC$ and $\Delta A'B'C'$ is the quadrilateral $CA'DA$. $\Delta ABC$ is partitioned into three triangles $\Delta CA'D$, $\Delta CDA$, and $\Delta A'DB$ with $\Delta CA'D$ and $\Delta CDA$ congruent. The triangles $\Delta CA'D$ and $\Delta A'DB$ having the same height at the vertex $D$ of bases $CA' = b$ and $A'B = a - b$. Since

$$b - (a - b) = 2b - a \geq b + c - a \geq 0,$$

Then the area of $\Delta A'DB$ is at most the area $\Delta CA'D$. Hence the sum of the areas of $\Delta CA'D$ and $\Delta CDA$ is at least $2/3$ of the area of $\Delta ABC$.

**Proof 2.** Assume $\angle A \geq \angle B \geq \angle C$. Let $L_1$ and $L_3$ be the two lines parallel to $BC$ and tri-secting the sides $AB$ and $AC$. Take $L = L_1$ as the reflection axis and the intersection is a hexagon (with choice of the angle above, otherwise this is not true). Now a direct computation of the areas three small triangles of the same height ($1/3$ of that of $\Delta ABC$ and a combined base of $BC$ shows that exactly $1/3$ areas of $\Delta ABC$ is not in the intersection.

4 Alice and Bob are playing a game. Alice thinks of a secret polynomial $p(x)$ with non-negative integer coefficients. Bob chooses an integer $a$ and Alice tells Bob the value $p(a)$. Then Bob chooses an integer $b$ and Alice tells Bob the value $p(b)$. Now Bob can tell Alice the polynomial $p(x)$ correctly. What is Bob's strategy for determining the polynomial $p(x)$?

**Solutions.** There are two solutions provided by contestants:

**Solution 1.** (Theoretical, but not practical). Bob will do the following:

(1) Take $a = 2$. Knowing the values $p(2)$, Bob will know the degree of $p(x)$ is at most $\log_2(p(2))$ and all coefficients of $p(x)$ is at most $p(2)$. Thus there is a finite number of possibility for $p(x)$. For all these finitely many possible polynomials, say, $\{p_1(x), \ldots, p_n(x)\}$ Bob will compute the solutions of $p_i(x) - p_j(x) = 0$ for all $1 \leq i < j \leq n$.

(2) Bob will choose an integer $b$ to not a solution of any of $p_i(x) - p_j(x) = 0$. Now knowing the answer $p(b)$, bob will compare it with all $p_i(b)$ and there will be only one that matches with Alice's answer and that has to be Alice's polynomial.

**Solution 2.** Very practical. Bob does not even need a calculator. As long as Alice can give the answer. Alice does not need a calculator either! Wait to hear this!!!

On Thursday, April 22, at the annual Friends of Mathematics Banquet, the winners played the game as Bob and a volunteer as Alice. After Alice wrote a polynomial $p(x)$ with a nonnegative coefficients in secret, Bob asked for the value $p(1)$ of the polynomial at 1. Then Bob asked the value $p(10^n)$ with $n$ be the smallest such that $p(1) < 10^n$. After knowing the value $p(10^n) = a_N 10^{nN} + a_{N-1}10^{n(N-1)} + \cdots + a_1 10 + a_0$ in decimal, Bob can tell value $a_0$ as the number which is the last $n-1$ decimal digits and the next $n$-digits form a number that is $a_1$ and so on to get the polynomial. The main point is to expand the value $p(10^n)$ in base $10^n$ and the coefficients $a_i$'s are the digits under this base.

More generally, suppose the polynomial is $p(x) = a_N x^N + a_{N-1}x^{N-1} + \cdots a_1 x + a_0$ with $a_i \geq 0$. Knowing $p(1) = a_N + \cdots + a_0$ one would get $a_i \leq p(1)$. then take $b > p(1)$ (thus $b > a_i$). The value

$$p(b) = a_N b^N + a_{N-1}b^{N-1} + \cdots + a_1 b + a_0$$

is the expression of $p(b)$ under base $b$. Since all the numbers here are integers, $a_0$ is the remainder of $p(b)$ divided by $b$, i.e., $q_0 = p(b) = q_1 b + a_0$ and $q_1 = q_2 b + a_1$, etc. Here $q_i$ is the quotient of $q_{i-1}$ when divided by $b$ with remainder $a_{i-1}$ for $i = 0, 1, \cdots$.

Using $b = 10^n$ will make the division much easier as answers are given in decimal form.