KANSAS STATE UNIVERSITY
Department of Mathematics

S. Thomas Parker Mathematical Competition
April 12, 2003

Instructions: Put your name on all papers you use and turn them all in. Try to solve as many problems as you can. For any problem you try, give as complete an answer as you can. Include a clearly written explanation of how you found your answer and why it is true. You may use drawings or calculations to help you for your justification, but your explanation should be convincing.

1. A cylinder is inscribed in a given right circular cone in such a manner as to maximize its volume.

   What is the ratio of the volume of the cylinder to the volume of the cone?

2. A prison with 64 cells is laid out in the same manner as the 64 squares on a chess-board. There are doors between any two adjacent cells. A prisoner in the upper left corner cell is told that if he can go through all of the cells exactly once and end up in the lower right corner cell, then he would be set free. Now help the prisoner determine whether it is possible for him to free himself. (He is super smart!) If possible, give the path he should follow. If not possible, give a proof.

3. Given any $n \geq 3$ and real numbers $x_1, x_2, \ldots, x_n$, define

$$f(x_1, x_2, \ldots, x_n) = 2(n-2)\sum_{i=1}^{n} x_i^2 - n(x_1 + x_2)^2$$

   (a) Prove that there are numbers $a_1, a_2, \ldots, a_n$ such that $f(a_1, a_2, \ldots, a_n) < 0$;

   (b) If $x_1 + x_2 + \cdots + x_n = 0$ then prove that $f(x_1, x_2, \ldots, x_n) \geq 0$.

4. David heard, on the phone, the following conversation taking place in a classroom between Jane, the math genius, and her English teacher Allen. "Jane, since you are good in math, here is a problem for you. I have three children and the product of their ages is 72 and the sum of the their ages is the room number of this room, can you tell me all my children's ages?" Apparently Jane went out and checked the room number and then said: "I need additional information." David heard Allen saying. "I have to go pick up the twins and take them to their older brother's soccer game!" Jane said, "Now I got it. But let's check with David on the other end of phone to see if he is smart enough to know the ages of the three Children."

   Help David figure out the ages with all information David heard on the phone? Note that David does not know the room number!