KANSAS STATE UNIVERSITY

Department of Mathematics

S. Thomas Parker Mathematical Competition
April 7, 2001

Instructions: Put your name on all papers you use and turn them all in. Try to solve as many problems as you can. For any problem you try, give as complete an answer as you can. Include a clearly written explanation of how you found your answer and why it is true. You may use drawings or calculations to help you for your justification, but your explanation should be convincing.

1. A car traveling on a straight line at 100 ft/sec hits its brakes and skids to a stop in 100 ft.

   a) Assuming that the deceleration of the car is constant (during its skid) determine how many seconds it takes for the car to come to a stop after first hitting its brakes.

   b) If instead the deceleration is inversely proportional to the speed that the car is traveling (in each instant), determine the time it takes to stop.

Solution a). Let $v(t)$ be the velocity function at time t seconds during the skid. Constant deceleration means that $v'(t) = -a$ for some constant $a > 0$. Thus $v(t) = 100 - at$ since $v(0) = 100$. The car will stop when $t = 100/a$. Let $s(t)$ be the distance the car has skidded during the first t seconds. Then

$$s(t) = \int_0^t v(x)dx = 100t - at^2/2.$$

We know that $s(100/a) = 100$, i.e.,

$$100 = 100 \cdot 100/a - a100^2/(2a^2) = 100^2/2a,$$

The solution for a is 50 and the car will stop when $t = 100/50 = 2$ seconds.

Solution b). Since the deceleration is inversely proportional to the speed of the car skidding, i.e., $a(t) = -k/v(t)$ for some $k > 0$. Since $a(t) = v'(t)$, then $-k = v(t)v'(t)$ and

$$-kt = \int_0^t v(x)v'(x)dx = \frac{1}{2}[v(t)^2 - v(0)^2] = \frac{1}{2}(v(t)^2 - 100^2).$$

Then $v(t) = \sqrt{100^2 - 2kt}$. Suppose the card will stop at T seconds, i.e., $0 = v(T) = \sqrt{100^2 - 2kT}$. Thus $T = \frac{100^2}{2k}$. Note that

$$100 = \int_0^T v(t)dt = \int_0^T \sqrt{100^2 - 2kt}dt = -\frac{1}{3k}[(100^2 - 2kT)^{\frac{3}{2}} - (100^2)^{\frac{3}{2}}] = \frac{100^3}{3k}.$$

Then $k = 100^2/3$ and $T = 3/2$ seconds.

2. For any positive integer n, set $(2n)!! = 2 \cdot 4 \cdot 6 \cdots (2n)$ and $(2n-1)!! = 1 \cdot 3 \cdot 5 \cdots (2n-1)$. Prove or disprove that for any positive integer n,

$$\frac{(2n-1)!!}{(2n)!!} \leq \frac{1}{\sqrt{2n+1}}.$$

Proof: First of all, the inequality is correct. One can either use induction on n or directly. Since each side is positive, by taking the square on both sides, we will to prove

$$\frac{1 \cdot 1 \cdot 3 \cdot 3 \cdot 5 \cdots (2n-1) \cdot (2n-1)}{2 \cdot 2 \cdot 4 \cdot 4 \cdot 6 \cdots (2n) \cdot (2n)} < \frac{1}{2n+1}.$$

The left hand side equals

$$\frac{1 \cdot 3}{2^2} \cdot \frac{3 \cdot 5}{4^2} \cdots \frac{(2n-3)(2n-1)}{(2n-2)^2} \cdot \frac{(2n-1)(2n+1)}{(2n)^2} \cdot \frac{1}{2n+1}.$$

Note that $\frac{(2i-1)(2i+1)}{(2i)^2} = \frac{(2i)^2-1}{(2i)^2} < 1$ for all $i = 1, 2, \ldots$. Now the inequality follows.

3. A 6 ft tall man is walking at a constant speed along a straight line away from a 30 ft high street lamp. At the instant that he is 100 ft from the base of the street lamp, his shadow reaches a vertical wall in front of him and begins to rise at a rate of one foot per second. How many more seconds after this instant does it take for him to reach the wall?

Solution. Let x be the distance between the man and the wall at the time when his shadow appears at the bottom of the wall. Then the distance between the lamp pole to the wall is $100 + x$. By similarity of the triangles we get $(100 + x)/30 = x/6$ which gives $100 + x = 5x$ or $x = 25$ feet.

After the man's shadow appears on the wall, we consider the first right triangles formed by the lamp, the man's head top, and the horizontal line between the man's head top and the lamp pole. The second right triangle formed by the top of the man's head, the shadow of the man's head top on the wall and the horizontal line at the height of the shadow of the head top. These two right triangles are similar. If $h(t)$ is the height of the shadow of the head top. Let v be the man's walking speed. Then vt is the distance the man traveled t seconds after his shadow appeared on the wall. Then we have

$$(100 + vt))/(30 - 6) = (25 - vt))/(6 - h(t)).$$

$$(6 - h(t))(100 + vt) = 24(25 - vt).$$

$$h(t) = 6 - 24(25 - vt)/(100 + vt).$$

Then $h'(t) = -24 \cdot \frac{-v(100+vt)-v(25-vt)}{(100+vt)^2}$. Since

$$1 = h'(0) = -24 \cdot \frac{-125v}{100^2} = \frac{3 \cdot 1000}{100^2} = \frac{3v}{10}$$

we get $v = 10/3$ and the time it takes for the man to reach to the wall is $25/v = 7\frac{1}{2}$ seconds.

4. An m by n checkerboard consisting of $m \cdot n$ identical unit squares is cut along one of its two diagonals. How many of the unit squares are cut into two pieces? Give a formula in terms of m and n.

Solution. Let us say that there are m rows and n columns such that $n \geq m$. The cut will cut at least one square in each column. However, when a cut crosses a horizontal border lines (the cracks between small squares) inside a column, two squares will be cut. There are exactly $n - 1$ crossings of the horizontal borders lines. If m and n are relatively prime, all $n-1$ crossings appear in side of a column so there are $m + n - 1$ squares being cut.

In general, let $d = \text{GCD}(m, n)$ and let $s = m/d$ and $t = n/d$, then the cut will go through the corner points at places (rs, rt) $(r = 0, 1, \ldots, d)$ where the cut crosses a horizontal border on inside any column. Inside the checker board, there are exactly $d - 1$ crossings not inside a column. Thus the formula is $n + m - \text{GCD}(, n, m)$.

Another way to look at the general situation is to divide checker board into $d \times d$ smaller boards of the size $s \times t$. The cut will go through d of the smaller boards. In each smaller boards $s + t - 1$ unit squares will be cut into two pieces. Thus there are $d(s + t - 1) = m + n - d$ unit squares being cut into two pieces.