KANSAS STATE UNIVERSITY

Department of Mathematics

S. Thomas Parker Mathematical Competition
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Instructions: Put your name on all papers you use and turn them all in. Try to solve as many problems as you can, in any order. For any problem you try, give as complete an answer as you can. Include a clearly written explanation of how you found your answer and why it is true. You may use drawings or calculations to help you for your justification, but your explanation should be convincing.

1. Compute the limit

$$\lim_{n\to\infty}\left(\frac{1}{2n+1}+\frac{1}{2n+2}+\frac{1}{2n+3}+\cdots+\frac{1}{3n}\right).$$

2. Two ships are each sailing in a straight line towards the point of intersection P of the two lines. The angle between their directions is $60°$. The first ship is currently ($t = 0$) a miles from P and travelling at a constant speed of u miles per hour, while the second ship is b miles from P and travelling at v miles per hour. On their present course, at what time $t \geq 0$, expressed in terms of u, v, a and b, will the ships be the closest?

3. A student is writing the numbers $1, 2, 3, \ldots, 10^{2000}$ in standard decimal form. Exactly how many times will she write the digit 1?

4. Suppose that a real valued function $f(x)$ defined on $(0, +\infty)$ is integrable on any interval $[a, b]$ with $0 < a < b$ and the improper integral $\int_a^\infty \frac{f(x)}{x}dx$ is convergent for every positive value a. Show that if $\lim_{x\to 0^+} f(x) = 1$, then the improper integral $\int_0^\infty \frac{f(x)}{x}dx$ is divergent. Also show that for any positive values α and β, the following improper integral is convergent and find its value.

$$\int_0^{+\infty}\frac{f(\alpha x)-f(\beta x)}{x}dx.$$