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Apollonian gasket, Descartes Theorem, conformal transformations and Minkowski space

(2 lectures)

We start with a simply looking geometric problem: Describe all configurations of four pairwise tangent circles on a plane. It turns out that the complete and clear solution of this problem uses tools from several different domains in mathematics. Moreover, the problem has natural many-dimensional analogues and requires a more precise and slightly modified formulation. We start with an elementary approach which immediately show us the necessity of refinements and modifications. First of all we must extend our definition of circles and include straight lines as circles of infinite radius. Second, we have to extend the plane by an infinite point, so that the plane becomes a closed surface, namely a two- dimensional sphere. Finally, we need to consider oriented circles, or pass from the circles to discs bounded by these circles. Note, that every circle on sphere bounds two discs, while oriented circle bounds a unique disc which remains on the left when we go along the boundary in the positive direction. So, the right problem is to describe all configuration of four discs on the sphere which are tangent (i.e. have exactly one common point) to each other. It turns out that many of these configurations have the property: the cur- vatures of the four discs (where curvature is defined as the quantity, reciprocal to the radius) are integers. It leads to the remarkable arithmetic problems, some of which are still unsolved. The detail exposition of the material form the second part of my future book "A tale on two fractals".