1 Some warm ups about groups

First recall the definition of a group:

Definition 1 *A* group *is a set G equipped with a constant e, a binary operation $\cdot$, and a unary operation $()^{-1}$ satisfying*

- $a \cdot (b \cdot c) = (a \cdot b) \cdot c$ *(associativity)*
- $a \cdot e = a = e \cdot a$ *(identity)*
- $a \cdot a^{-1} = e = a^{-1} \cdot a$ *(inverse)*

Exercise 2 *Prove that in any group e is unique, in the sense that if any element x satisfies $a \cdot x = a = x \cdot a$ for all $a \in G$, then $x = e$.*

Definition 3 *If G and H are groups, a* group homomorphism *from G to H is a function $f : G \to H$ such that*

- $f(e) = e$
- $f(a \cdot b) = f(a) \cdot f(b)$

Exercise 4 *Prove that if $f : G \to H$ is a group homomorphism, then $f(a^{-1}) = f(a)^{-1}$ for every element $a \in G$.*

Exercise 5 *The kernel of a group homomorphism $f : G \to H$ is the set of all elements $x \in G$ such that $f(x) = e$. Prove that the kernel is a subgroup of G (i.e. a subset which is closed under the operations, including containing e). Prove moreover that it is a normal subgroup. (This means that whenever x is in the subgroup, and y is any element of G (not just the subgroup), then yxy^{-1} is an element of the subgroup.)*

2 Quandles, Fundamental Quandles and Knot Quandles

Quandles were originally introduced by Joyce as an algebraic invariant of classical knots and links. They may be regarded as an abstraction from groups inasmuch as some of the most important examples arise by considering a group with left and right conjugation as operations.

Definition 6 *A* quandle *is a set Q equipped with two binary operations $\triangleright$ and $\underline{\triangleright}$ satisfying*

$$\begin{array}{ll} \forall x \in Q & x \triangleright x = x \\ \forall x, y \in Q & (x \triangleright (x \underline{\triangleright} y) = y = (x \underline{\triangleright} (x \triangleright y) \\ \forall x, y, z \in Q & (x \triangleright (y \triangleright z) = (x \triangleright y) \triangleright (x \triangleright z) \end{array}$$

Algebraic structures satisfying the second and third axioms only have been studied under the name "racks" by Fenn and Rourke. Structures satisfying the third axiom only are called "left distributive semigroups" by universal algebraists.

Examples abound:

Example 7 *If G is any group, we can make G into a quandle by letting $x \triangleright y = xyx^{-1}$ and $x \underline{\triangleright} y = x^{-1}yx$. Likewise any union of conjugacy classes in a group G forms a subquandle.*

This example is of particular importance for the theory of quandles, as a representation theorem due to Joyce [**?**] shows that all free quandles embed into (free) groups as a disjoint union of conjugacy classes, and thus that the universally quantified equations holding in all quandles are precisely those holding in all quandles of this form.

Exercise 8 *Prove the theorem of Joyce cited in the previous paragraph.*

Example 9 *Given an R-linear automorphism T of an R-module V, V becomes a quandle with $x \triangleright y = T(y-x) + x$ and $x \underline{\triangleright} y = T^{-1}(y-x) + x$.*

Exercise 10 *Verify that the operations just given make the R-module V into a quandle. (If you don't know about modules, do this in the special case where V is a vector space (say over the real numbers) and T is an invertible linear transformations.)*

Exercise 11 *Let X be a real vector space equipped with an antisymmetric bilinear form $\langle -, - \rangle : X \times X \to \mathbb{R}$.*

Show that X is a quandle when equipped with the operations

$$x \triangleright y = y + \langle x, y \rangle x$$

and

$$x \underline{\triangleright} y = y - \langle x, y \rangle x$$

We call such a quandle a (real) alternating quandle.

Exercise 12 *Given an alternating quandle structure on V show that there is a quandle structure on the space of orbits of the action of the multiplicative group $\{1, -1\}$ on V by scalar multiplication with operations induced by those on V. (The orbits are equivalence classes where v is equivalent to $-v$.)*

Joyce's principal motivation in considering this structure was to provide an algebro-topological invariant of classical knots more sensitive than the fundamental group of the complement.

We can consider the corresponding notion in arbitrary dimensions. We consider pairs of a space and a subspace, equipped with a point in the complement of the subspace (X, S, p). In particular we consider the "noose" or "lollipop": $(N, \{0\}, 2)$ where N is the subspace of $\mathbb{C}$ consisting of union of the unit disk and the line segment $[1, 2]$ in the real axis.

By a map of pointed pairs we mean a continuous map which preserves the base point and both the subspace and its complement. We can then make

Definition 13 *The* fundamental quandle $\Pi(X, S, p)$ *of a pointed pair* (X, S, p) *is the set of homotopy classes of maps of pointed pairs (where homotopies are through maps of pointed pairs), equipped with the operations* $x \triangleright y$ *(resp.* $x \underline{\triangleright} y$*) induced by appending the path from the base point obtained by traversing* $x([1, 2])$*, followed by* $x(S^1)$ *oriented counterclockwise (resp. clockwise), followed by traversing* $x([1, 2])$ *in the opposite direction to the path* $y([1, 2])$ *and reparametrizing.*

In the case where both the space and its subspace are smooth oriented manifolds and the subspace is of codimension 2, it is possible to identify a particularly interesting subquandle of the fundamental quandle.

Definition 14 *The* knot quandle $Q(M, K, p)$ *of a pointed pair* (M, K, p)*, where* M *is a smooth manifold,* K *a smooth embedded submanifold of codimension 2 is the subquandle of* $\Pi(M, K, p)$ *consisting of all maps of the noose such that the bounding* S^1 *has linking number* 1 *with* K*. (Note: this is in the signed sense.)*

Joyce showed that the knot quandle of a classical knot determined the knot up to orientation.

It is easy to see that there is a relationship between the knot quandle $Q(\Sigma, S, p)$ and the fundamental group of the complement $\pi_1(\Sigma \setminus S, p)$: an action of $\pi_1(\Sigma \setminus S, p)$ on $Q(\Sigma, S, p)$ by quandle homomorphisms is given by appending a loop representing an element of π_1 to the initial path of the noose and rescaling.

There is, however, an more intimate relationship between them:

Definition 15 *(Joyce) An* augmented quandle *is a quadruple*

$$(Q, G, \ell : Q \to G, \cdot : Q \times G \to Q)$$

where Q *is a quandle,* G *is a group,* $\cdot$ *is a left-action of* G *on* Q *by quandle homomorphisms, and the set-map* ℓ *(called the* augmentation*) satisfies*

$$\begin{aligned} \ell(q) \cdot q &= q \\ \ell(\gamma \cdot q) &= \gamma \ell(q) \gamma^{-1} \end{aligned}$$

If you haven't met the fundamental group or homotopy in your studied, you can skip the proofs below (as they won't be at all edifying).

Proposition 16 *For any oriented manifold* M *with an oriented, properly embedded codimension 2 submanifold* K *and a point* $p \in M \setminus K$*, the quadruple*

$$(Q(M, K, p), \pi_1(M \setminus K, p), \ell, \cdot)$$

where $\cdot$ *is the action described above, and* $\ell(q)$ *is the homotopy class of the loop at* p *which traverses the arc, then the boundary of the disk counterclockwise, then the arc back to* p*, is an augmented quandle. We call the loop at* p *just described as a representative for* $\ell(q)$ *the* canonical loop *of the noose* q*.*

proof: Having noted that the action of $\pi_1(M \setminus K, p)$ is by quandle homomorphisms (a fact which follows essentially by conjugation in the fundamental groupoid—the reader may fill in the details), it remains only to verify that the map ℓ satisfies the two conditions specified in the definition of augmented quandles.

The first reduces to the idempotence of the quandle operation. The second follows from the fact that the appended loop occurs twice in the specification of $\ell(q \cdot \gamma)$, initially in the outgoing arc from p with positive orientation, and again in the incoming arc to p with reversed orientation.

We also have

Proposition 17 *If M is simply connected, $\ell(Q(M, K, p))$, the image of the augmentation, generates $\pi_1(M \setminus K, p)$.*

proof: This follows from van Kampen's Theorem: killing all of the noose boundaries kills the fundamental group, and thus the noose boundaries generate.

3 More exercises with quandles (some open-ended)

Exercise 18 *Define a quandle homomorphism.*

Exercise 19 *If $f : Q \to R$ is a quandle homomorphism, what structure does its image have? What structure does the inverse image of an element have?*

We saw that a group can be made into a quandle by using conjugation as the operations. Joyce denoted this quandle by $Conj(G)$.

We also had defined involutory quandles to be sets with a single operation $\triangleright$ which were quandles with $\triangleright = \underline{\triangleright}$

Exercise 20 *Show that is G is a group, then the operation $x \triangleright y = xy^{-1}x$ makes G into an involutory quandle. (Joyce denoted this quandle $Core(G)$.*

Exercise 21 *Write tables of operations for $Conj(G)$ and $Core(G)$ for some small groups.*

Exercise 22 *What can you say about how subquandles are related to the quandle containing them? To each other?*