

## REU – PROJECT 2 – JUNE 2005

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### 2. GROUND EFFECT

The goal here is to follow the treatment in Fisher's *Complex Variables* p. 246-259, of the Jukowski airfoil, with the additional requirement that the ground, i.e. a horizontal line, is also a streamline.

So let  $\mathbb{H} = \{z : \text{Im } z > 0\}$ , and let  $D$  be a disk such that  $i \in D \subset \mathbb{H}$ . Actually, for  $0 < r < 1$ , we will let

$$D = D_r = \left\{ z \in \mathbb{H} : \left| \frac{z - i}{z + i} \right| \leq r < 1 \right\}$$

Check that the circles  $\partial D_r$  foliate the upper half-plane as  $r$  varies between 0 and 1.

We want to find an analytic and one-to-one map  $F$  of the domain  $\Omega = \mathbb{H} \setminus D_r$  onto the domain  $\Sigma = \mathbb{H} \setminus L_t$ , where

$$L_t = \{z = x + i \in \mathbb{H} : x \in [-t, t]\}$$

Given  $r$  there is only one value of  $t$  for which such a map exists.

We will first compute the harmonic function  $\psi = \text{Im } F$ . Notice that  $\psi = 0$  on  $\mathbb{R}$ ,  $\psi = 1$  on  $\partial D_r$ ,  $\psi > 0$  on  $\Omega$ , and tends to  $\infty$  at  $\infty$ . We give  $\psi$  as the sum of two functions:  $\psi = \omega + ch$ , for some  $c > 0$ . The function  $\omega$  is the unique bounded harmonic function which solves the Dirichlet Problem on  $\Omega$  with 0 on  $\mathbb{R}$  and 1 on  $\partial D_r$ . The function  $h$  is the unique (up to positive multiple) positive harmonic function on  $\Omega$  which is 0 on  $\mathbb{R}$  and on  $\partial D_r$ . In order to write down formulas for these two functions we change variables and apply the linear fractional transformation  $(z - i)/(z + i)$  which sends  $\mathbb{H}$  onto  $\mathbb{D} = \{|z| < 1\}$ , and sends  $\Omega$  to the annulus  $\tilde{\Omega} = \{r < |w| < 1\}$ . In these new variables the function

$$\tilde{\omega}(w) = \frac{\log |w|}{\log r}$$

is harmonic, is  $= 0$  on  $\{|w| = 1\}$  and is  $= 1$  on  $\{|w| = r\}$ . So

$$\omega(z) = \tilde{\omega} \left( \frac{z - i}{z + i} \right).$$

Likewise,

$$h(z) = \tilde{h} \left( \frac{z - i}{z + i} \right)$$

where  $\tilde{h}$  is positive harmonic on the annulus  $\tilde{\Omega}$ , is  $= 0$  on  $\partial\tilde{\Omega} \setminus \{1\}$ , but tends to infinity as  $w$  tends to  $1$  from within  $\tilde{\Omega}$ . Such function is known as the **Poisson kernel** at  $1$  for the annulus  $\tilde{\Omega}$ . A formula is given in Henrici's book Vol. III, p. 229-230.

The constant  $c$  in  $\psi = \omega + ch$ , cannot be chosen arbitrarily. It is the unique choice of  $c$  which will make the period of  $\psi$  around  $D_r$  equal to zero. The **period** of  $\psi$  is defined to be

$$\int_{\partial D_r} \frac{\partial \psi}{\partial \eta} ds$$

where  $\partial/\partial\eta$  is the outer normal derivative, i.e. the period of  $\psi$  is the circulation of the flow around  $D_r$ .

Several problems arise now. Can you draw the flow? Where are the stagnation points? Can you add a circulation term so as to create lift? If so will the lift depend on the distance to the ground (ground effect)? What does the airfoil look like when you apply  $F$  to a circle that is not necessarily equal to  $\partial D_r$ , but goes through one of the stagnation points?