# REU PROJECT KANSAS STATE UNIVERSITY, SUMMER 2005
# THE SMALE CONJECTURE

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## 1. Smale's conjecture

**Definition 1.1.** Given a polynomial $p(z) = b_0 + b_1 z + b_2 z^2 + \cdots + b_d z^d$, we say that $z$ is a critical point of $p$ if $p'(z) = 0$, where $p'(z) = b_1 + 2b_2 z + \cdots + d b_d z^{d-1}$ is the derivative of $p$.

**Conjecture 1.2** (Smale). *Suppose $p$ is a polynomial of degree $d$, with $p(0) = 0$ and $p'(0) = 1$. Prove that there is a critical point $z$ so that*

$$\frac{|p(z)|}{|z|} \leq 1 - \frac{1}{d}$$

For any complex non-zero number $c$ we can replace the polynomial $p(z)$ by $cp(z/c)$. This transformation preserves the properties $p(0) = 0$ and $p'(0) = 1$, but allows to change one higher order (non-zero) coefficient $b_j$ ($j = 2, \ldots, d$), of our choice, to any other prescribed non-zero complex number.

### 1.1. **Binomials.**

The case of binomials is well known and is conjectured to be the extremal case. Following the discussion above, without loss of generality

$$p(z) = z - \frac{1}{d} z^d$$

Then $p'(z) = 1 - z^{d-1}$, hence the critical points of $p$ are the $(d-1)$-roots of unity $\omega_k = e^{i2\pi k/(d-1)}$ ($k = 0, \ldots, d-2$) and satisfy

$$p'(\omega_k) = 1 - \omega_k^{d-1} = 0.$$

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So

$$\frac{p(\omega_k)}{\omega_k} = 1 - \frac{1}{d}\omega_k^{d-1} = 1 - \frac{1}{d}.$$

1.2. **Trinomials.** The next natural case is that of trinomials, which doesn't seem to have been treated in the literature. Namely, we consider polynomials of the form $p(z) = z + b_k z^k + b_d z^d$, with $1 < k < d$. Without loss of generality, we can assume that

$$p(z) = z + az^k - \frac{1}{d}z^d.$$

Its critical points solve the equation

$$p'(z) = 1 + akz^{k-1} - z^{d-1} = 0 \tag{1.2.1}$$

Therefore, any critical point $\zeta$ of $p$ satisfies

$$a\zeta^{k-1} = -\frac{1}{k} + \frac{1}{k}\zeta^{d-1}$$

We want to estimate

$$\frac{p(\zeta)}{\zeta} = 1 + a\zeta^{k-1} - \frac{1}{d}\zeta^{d-1} = 1 - \frac{1}{k} + \left(\frac{1}{k} - \frac{1}{d}\right)\zeta^{d-1}$$

However, the product of all solutions to (1.2.1) is equal to one in modulus. So we can always choose one such root with $|\zeta| \leq 1$. Then

$$\frac{|p(\zeta)|}{|\zeta|} \leq 1 - \frac{1}{k} + \frac{1}{k} - \frac{1}{d} = 1 - \frac{1}{d}, \tag{1.2.2}$$

i.e., Smale's conjecture holds in this case. We can actually say more in this case. Namely, trinomials also satisfy the stronger Tischler Conjecture (which has now been disproved for arbitrary polynomials, see [Tys05]). This conjecture says that there is a critical point $z$ such that

$$\left|\frac{p(z)}{z} - \frac{1}{2}\right| \leq 1 - \frac{1}{d}.$$

This follows because $k \geq 2$, hence

$$\frac{p(\zeta)}{\zeta} - \frac{1}{2} = \frac{1}{2} + a\zeta^{k-1} - \frac{1}{d}\zeta^{d-1} = \frac{1}{2} - \frac{1}{k} + \left(\frac{1}{k} - \frac{1}{d}\right)\zeta^{d-1}$$

which yields

$$\left|\frac{p(\zeta)}{\zeta} - \frac{1}{2}\right| \leq \frac{1}{2} - \frac{1}{k} + \frac{1}{k} - \frac{1}{d} = \frac{1}{2} - \frac{1}{d}. \tag{1.2.3}$$

Moreover, equality holds in either (1.2.2) or (1.2.3) if and only if $\zeta^{d-1} = 1$, hence by (1.2.1), if and only if $ak\zeta^{k-1} = 0$, which happens if and only if $a = 0$ (since $|\zeta| = 1$), and therefore if and only if $p$ is a binomial to start with.

1.3. **5th degree polynomials.** Consider a 5th degree polynomial of the form $p(z) = z + \frac{b_1}{2}z^2 + \frac{b_3}{4}z^4 - \frac{1}{5}z^5$, and $p'(z) = -\prod_{j=1}^{4}(z - \theta_j)$ where the $\theta_j$s are the critical points of the polynomial. Then

$$S(\theta_1) = \frac{|p(\theta_1)|}{|\theta_1|} = |1 + b_1\theta_1 + b_3\theta_1^3 - \frac{1}{5}\theta_1^4|$$

Solving $p' = 0$ for $b_1$ and substituting gives

$$S(\theta_1) = \frac{1}{2} - \frac{1}{4}b_3\theta_1^3 + \frac{1}{5}\theta_1^4$$

Note that $b_3$ is the sum of the critical point of $p$, therefore

$$S(\theta_1) = \frac{1}{2} + \frac{1}{20}\theta_1^4 - \frac{1}{4}\theta_1^3(\theta_2 + \theta_3 + \theta_4)$$

We then note that

$$\begin{aligned}
\sum_{j=1}^{4} \frac{|S(\theta_j) - \frac{1}{2}|}{|\theta_j^3|} &\leq \left(\frac{1}{20} + \frac{3}{4}\right)(|\theta_1| + |\theta_2| + |\theta_3| + |\theta_4|) \\
&\leq \left(1 - \frac{1}{5}\right)\sum_{j=1}^{4} \frac{1}{|\theta_1 \cdots \hat{\theta}_j \cdots \theta_4|} \\
&\leq \left(1 - \frac{1}{5}\right)\sum_{j=1}^{4} \frac{1}{|\theta_j|^3}
\end{aligned}$$

$$\frac{1}{\sum_{j=1}^{4}\frac{1}{|\theta_j|^3}} \sum_{j=1}^{4} \frac{|S(\theta_j) - \frac{1}{2}|}{|\theta_j^3|} \leq 1 - \frac{1}{5}$$

This equation is a weighted average of the quantities $|S(\theta_j - 1/2|$ with weights $1/|\theta_j|^3$. If the average is $\leq 1 - \frac{1}{5}$, then at least one $\theta_j$ satisfies $|S(\theta_j - 1/2| \leq 1 - \frac{1}{5}$. However, this is not a good enough bound.

1.4. **New polynomials from Old polynomials.** Let $f(z)$ and $g(z)$ be be polynomials satisfying the conditions of Smale's Conjecture, and let $f$ have some critical point $z_j$ such that $f(z_j) = f'(z_j) = 0$. From $f$ and $g$, one can construct the following polynomials satisfying Smale's Conjecture:

$$p(z) = f(z)g'(z)$$
$$p(z) = \frac{f(z)g(z)}{z}$$
$$p(z) = g(f(z))$$

We begin by noting that $p(z) = 0$ for each equation when evaluated at the point $z_j$. Finding $p'$ establishes that a critical point exists for each polynomial $p$ at the point $z_j$:

$$p'(z) = f'(z)g'(z) + f(z)g''(z)$$
$$p'(z) = \frac{z(f'(z)g(z) + f(z)g'(z)) - f(z)g(z)}{z^2}$$
$$p'(z) = g'(f(z))f'(z)$$

Due to the extra constraints placed on $f(z)$, each of the derivatives is 0 at the point $z_j$, thus each polynomial satisfies Smale's conjecture.

Given a set of polynomials that satisfy Smale's Conjecture at a common point $z_j$, one can also state that the mean of these polynomials satisfies Smale's Conjecture. Let $f_1(z), \cdots, f_n(z)$ be the given series of polynomials, where $d_{f_1} \geq d_{f_2} \geq \cdots \geq d_{f_n}$. Then

$$p(z) = \frac{f_1(z) + \cdots + f_n(z)}{n}$$
$$p'(z_j) = \frac{f_1'(z_j) + \cdots + f_n'(z_j)}{n} = 0$$

Evaluating p at $z_j$, we get

$$\frac{|p(z_j)|}{|z_j|} = \frac{1}{n}\frac{|(f_1(z_j) + \cdots + f_n(z_j))|}{|z_j|} \leq \frac{1}{n}((1 - \frac{1}{d_{f_1}}) + \cdots + (1 - \frac{1}{d_{f_n}}))$$
$$= 1 - \frac{1}{n}(\frac{1}{d_{f_1}} + \cdots + \frac{1}{d_{f_n}})$$

Examining the fractional part shows that

$$\frac{1}{d_{f_1}} + \cdots + \frac{1}{d_{f_n}} \geq \frac{1}{d_{f_1}} + \cdots + \frac{1}{d_{f_1}} = \frac{n}{d_{f_1}}$$

Therefore

$$\frac{|p(z_j)|}{|z_j|} \leq 1 - \frac{1}{n}(\frac{1}{d_{f_1}} + \cdots + \frac{1}{d_{f_n}}) \leq 1 - \frac{1}{n}\frac{n}{d_{f_1}} = 1 - \frac{1}{d_{f_1}}$$

Where $d_{f_1}$ is the degree of the polynomial $p$.

1.5. **Smale Value Bounds.** As will be derived in (2), the Smale values can be expressed in terms of the critical points of the polynomial. The Smale values take the form of

$$\frac{|p(z_1)|}{|z_1|} = \left| \left(1 - \frac{1}{2}\right) - \left(\frac{1}{2} - \frac{1}{3}\right)(A + B + C + \cdots) + \left(\frac{1}{3} - \frac{1}{4}\right)(AB + AC + \cdots) - \cdots \right|$$

$$\leq \left(1 - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right)|A + B + C + \cdots| + \cdots$$

Where $A = \frac{z_1}{z_2}$, $B = \frac{z_1}{z_3}$, etc. Assume that $z_1 \leq z_2 \leq \cdots \leq z_{d-1}$, then all $A, B, \cdots, AB, AC, \cdots$ will be $\leq 1$ for the Smale value at $z_1$. Therefore

$$\leq \left(1 - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right)(1 + 1 + 1 + \cdots) + \cdots$$

Then we need only find the numbers of terms composed of $A, B, AB, \cdots$ to find an upper bound for the Smale value at $z_1$. Therefore, the upper bound for the Smale value is

$$\sum_{j=0}^{d-2} \left[ \left( \frac{1}{j+1} - \frac{1}{j+2} \right) \frac{(d-2)!}{(d-2-j)!j!} \right]$$

Unfortunately, this upper bound grows rapidly. However, it does reveal bounds for polynomials of degrees 5, 6, and 7: $\frac{13}{10}, \frac{19}{10}, \frac{20}{7}$, respectively.

## 2. Numerical experiments

With the above notations $p'(z) = 1 + 2b_2 z + \cdots d b_d z^{d-1}$ is a polynomial of degree $d-1$ and hence, by the Fundamental Theorem of Algebra, it has exactly $d-1$ complex zeros: $z_1, \ldots, z_{d-1}$. By factoring out the roots we can write

$$p'(z) = \prod_{j=1}^{d-1} (1 - a_j z)$$

where $a_j = 1/z_j$ (note that the $z_j$'s are not zero). Fix $k = 1, 2, \ldots, d-1$. By the Fundamental Theorem of Calculus,

$$p(z_k) = \int_{[0,z_k]} p'(z) dz$$

where $[0, z_k]$ is the straight segment from 0 to $z_k$. Parametrize this segment by letting $z = tz_k = t/a_k$, so that $dz = dt/a_k$. Then

$$p(z_k) = z_k \int_0^1 (1-t) \prod_{j=1, j\neq k}^{d-1} \left(1 - t\frac{a_j}{a_k}\right) dt$$

Unfolding the product gives

$$\prod_{j=1, j\neq k}^{d-1} \left(1 - t\frac{a_j}{a_k}\right) = \sum_{n=0}^{d-2} c_n t^n \tag{2.0.4}$$

where $c_0 = 1$ and the other coefficients $c_n$ depend on the ratios $a_j/a_k$. Multiplying this sum by $(1-t)$ and integrating $dt$ from 0 to 1, we get

$$p(z_k) = z_k \sum_{n=0}^{d-2} c_n \left(\frac{1}{n+1} - \frac{1}{n+2}\right)$$

We would like the following inequality to hold

$$\left|\sum_{n=0}^{d-2} c_n \left(\frac{1}{n+1} - \frac{1}{n+2}\right)\right| \leq 1 - \frac{1}{d}. \tag{2.0.5}$$

If we had $|c_n| \leq 1$, then by the triangle inequality

$$\begin{aligned}
\left|\sum_{n=0}^{d-2} c_n \left(\frac{1}{n+1} - \frac{1}{n+2}\right)\right| &\leq \sum_{n=0}^{d-2} \left(\frac{1}{n+1} - \frac{1}{n+2}\right) \\
&= \left(1 - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \cdots \left(\frac{1}{d-1} - \frac{1}{d}\right) \\
&= \left(1 - \frac{1}{d}\right)
\end{aligned}$$

which is what we want. However, the coefficients $c_j$ are not necessarily less than 1 in modulus. So by letting $k$ vary from 1 to $d-1$, we get $d-1$ inequalities for the $c_j$'s, and the conjecture says that at least one of these is satisfied.

**Case d** $= 4$. Let $k = 1$. Then the product in (2.0.4) is

$$\left(1 - t\frac{a_2}{a_1}\right)\left(1 - t\frac{a_3}{a_1}\right) = 1 - (A+B)t + ABt^2$$

where $A = a_2/a_1$ and $B = a_3/a_1$. Without loss of generality $AB = -1$, so given $a_1$ and $a_2$ we can set $a_3 = -a_1^2/a_2$. The inequality corresponding to $a_k = a_1$ is

$$\left|\left(1 - \frac{1}{2}\right) - (A+B)\left(\frac{1}{2} - \frac{1}{3}\right) + AB\left(\frac{1}{3} - \frac{1}{4}\right)\right| \leq 1 - \frac{1}{4}$$

When $a_k = a_2$ we have

$$\left|\left(1-\frac{1}{2}\right)-\frac{B+1}{A}\left(\frac{1}{2}-\frac{1}{3}\right)+\frac{B}{A^2}\left(\frac{1}{3}-\frac{1}{4}\right)\right| \leq 1-\frac{1}{4}$$

When $a_k = a_3$ we have

$$\left|\left(1-\frac{1}{2}\right)-\frac{A+1}{B}\left(\frac{1}{2}-\frac{1}{3}\right)+\frac{A}{B^2}\left(\frac{1}{3}-\frac{1}{4}\right)\right| \leq 1-\frac{1}{4}$$

**PROJECT:** Write a program which takes as graphic input three non-zero complex numbers $a_1, a_2, a_3 = -a_1^2/a_2$ and highlights the one for which the corresponding inequality is satisfied. Also allow the possibility to drag the $a_j$'s around with the mouse and see which one gives the inequality. Repeat for higher degrees.

2.1. **Verifying Smale's conjecture(by computation):** As a first pass to trying to prove Smale's conjecture, we wrote a program that took a 4-degree user-defined polynomial, which was defined by its critical points and Smale's hypotheses, and sequentially computed the inequalities of Smale and output which ones held. Figure 1 shows the Maple v9.5 source code as well as an example.

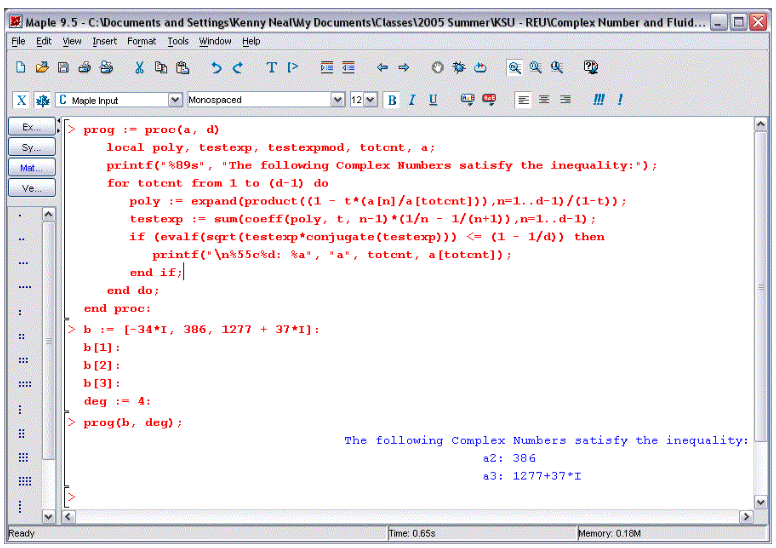


Figure 1.

**1st Edit(Matrix Version):** The second version of our program added the ability to move one of the points around. Rather than using a user-based graphical plot movement technique, we decided to integrate a systematical movement system into the program where a single point was moved around a chosen region with a chosen increment. The result of these methods was a matrix with components corresponding to the satisfied inequalities, at positions corresponding to the current position of the volatile point. Figure 2 shows an example of a typical Maple output.

$$\begin{bmatrix}
r_1 r_2 r_3 r_4 & r_1 r_2 r_3 r_4 & r_1 r_2 r_3 & r_1 r_2 r_3 r_4 & r_2 r_3 r_4 & r_1 r_2 r_3 r_4 & r_1 r_2 r_3 r_4 & r_1 r_3 & r_1 r_2 r_3 \\
r_1 r_2 r_3 r_4 & r_1 r_2 r_3 r_4 & r_1 r_2 r_3 r_4 & r_2 r_3 r_4 & r_1 r_2 r_3 r_4 & r_2 r_3 r_4 & r_1 r_2 r_4 & r_1 r_3 r_4 & r_1 r_2 r_3 r_4 \\
r_1 r_2 r_3 r_4 & r_2 r_3 r_4 & r_1 r_2 r_3 r_4 & r_1 r_2 r_3 r_4 & r_2 r_4 & r_1 r_2 r_3 & r_2 r_3 r_4 & r_3 r_4 & r_1 r_3 r_4 \\
r_2 r_3 r_4 & r_2 r_3 r_4 & r_2 r_3 r_4 & r_2 r_3 r_4 & r_2 r_3 r_4 & r_2 r_3 r_4 & r_2 r_3 & r_2 r_3 r_4 & r_1 r_2 r_3 r_4 \\
r_1 r_2 r_4 & r_3 r_4 & r_2 r_3 r_4 & r_2 r_3 & \mathit{Origin} & r_3 r_4 & r_2 r_4 & r_1 r_2 r_3 r_4 & r_1 r_3 r_4 \\
r_1 r_3 r_4 & r_2 r_4 & r_2 r_3 r_4 & r_1 r_2 r_3 r_4 & r_2 r_3 r_4 & r_3 r_4 & r_2 r_3 r_4 & r_1 r_2 r_3 r_4 & r_1 r_2 r_4 \\
r_2 r_3 r_4 & r_3 r_4 & r_2 r_3 r_4 & r_2 r_3 r_4 & r_2 r_3 r_4 & r_3 r_4 & r_2 r_3 r_4 & r_1 r_2 r_3 r_4 & r_2 r_4 \\
r_2 r_3 r_4 & r_1 r_2 r_3 r_4 & r_1 r_2 r_3 r_4 & r_3 r_4 & r_2 r_3 r_4 & r_3 r_4 & r_1 r_2 r_3 r_4 & r_2 r_3 r_4 & r_1 r_2 r_3 r_4 \\
r_1 r_2 r_3 r_4 & r_1 r_2 r_3 & r_1 r_2 r_3 r_4 & r_1 r_3 r_4 & r_1 r_4 & r_1 r_2 r_3 r_4 & r_1 r_2 r_3 & r_1 r_2 r_3 r_4 & r_1 r_2 r_3 r_4
\end{bmatrix}$$

$$\begin{bmatrix}
-4+4I & -3+4I & -2+4I & -1+4I & 4I & 1+4I & 2+4I & 3+4I & 4+4I \\
-4+3I & -3+3I & -2+3I & -1+3I & 3I & 1+3I & 2+3I & 3+3I & 4+3I \\
-4+2I & -3+2I & -2+2I & -1+2I & 2I & 1+2I & 2+2I & 3+2I & 4+2I \\
-4+I & -3+I & -2+I & -1+I & I & 1+I & 2+I & 3+I & 4+I \\
-4 & -3 & -2 & -1 & \mathit{Origin} & 1 & 2 & 3 & 4 \\
-4-I & -3-I & -2-I & -1-I & -I & 1-I & 2-I & 3-I & 4-I \\
-4-2I & -3-2I & -2-2I & -1-2I & -2I & 1-2I & 2-2I & 3-2I & 4-2I \\
-4-3I & -3-3I & -2-3I & -1-3I & -3I & 1-3I & 2-3I & 3-3I & 4-3I \\
-4-4I & -3-4I & -2-4I & -1-4I & -4I & 1-4I & 2-4I & 3-4I & 4-4I
\end{bmatrix}$$

Figure 2.

**2nd Edit(Product Inequality):** After a prolonged search for a counterexample to Smale's conjecture using the above program, the validity of the product inequality was explored by replacing lines 42 and 43 with the following code:

```
42 if (ceil(product(evalf(sqrt(testexp[k]*conjugate(testexp[k]))),k=1..d-1))=1) then
43 johnners:=location/yes;
44 else
45 johnners:=NO!;
```

**Product Inequality Conjecture:** Given any polynomial whose critical points all lie within a quadrant in the complex plane, the product inequality (shown below), where $z_j$ is a critical point, holds true.

$$\left| \prod_j \frac{p(z_j)}{z_j} \right| \leq 1$$

## References

[Tys05] Jeremy T. Tyson.
Counterexamples to Tischler's strong form of Smale's mean value conjecture.
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