CONVEX, SYMMETRIC BODIES UNIQUELY DEFINED BY THEIR CHORD FUNCTION.

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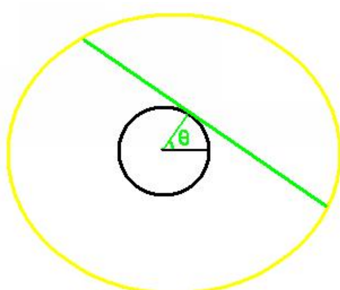
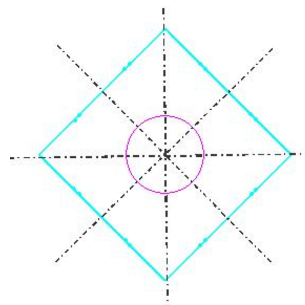
Let the chord function of the body, $c(\theta) : [0, 2\pi) \to \mathbb{R}^2$ be the lenght of the chord tangent to the body at θ. For instance, $c(\theta)$ in the body below (left) would be the length of the green line segment.





In this paper, a symmetric body is absolutely symmetric, i.e. neither reflection along an axis nor rotation by $\frac{\pi}{2}$ changes the body. A symmetric body is its own reflection along $y = 0$, $x = 0$, $y = x$, and $y = -x$, as shown by the diamond above (center).

A symmetric chord function means $c(\theta) = c\left(\frac{\pi}{2} - \theta\right) = c\left(\theta + \frac{\pi}{2}\right)$.

For simplicity, unless stated otherwise we are chosing our units so that the inner circle has radius 1, and is centered at the origin.

Lemma 1. *If two symmetric bodies have the same chord function, then for $n \in \mathbb{Z}$, $0 \le n < 8$ the endpoints of the chords at $\theta = \frac{\pi}{4}n$ are on both bodies.*

Proof. Consider any symmetric, convex body. For n an integer and $\theta = \frac{\pi}{4}n$, the chord at θ touches the circle at its midpoint. For instance, when $\theta = 0$ the tangent chord is vertical, so its endpoints are reflections of each other along $y = 0$. The endpoints of these eight chords must be on the boundary any other symmetric body with the same chord function, because the chords are the same length and have the same midpoint for $\theta = \frac{\pi}{4}n$. $\square$

For any body, there is exactly one radius, $r^* = \frac{c(0)}{2+2\sqrt{2}}$, of the inner circle for which the right endpoint (as viewed from inside the circle) of the chord at $\theta = \pi$ is the same as the left endpoint of the chord at $\theta = \frac{\pi}{4}$, and so the eight chords will form an amish star and the endpoints of those chords will only generate eight points on the boundary of the body rather than sixteen.

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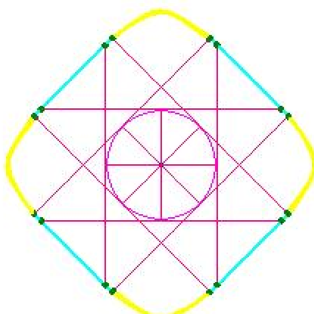
If $c(0) \neq 2$ or $c(\frac{\pi}{4}) \neq 2$, then for any other radius, at least twelve points are generated from the chords at $\theta = \frac{\pi}{4}n$.

Part 1

Consider a symmetric, convex body A for which the line passing through the left endpoint of the chord at $\theta = 0$ and the right endpoint of the chord at $\theta = \frac{3\pi}{4}$ has slope -1 or 0. This is basically a diamond or square with the corners possibley smoothed a bit.

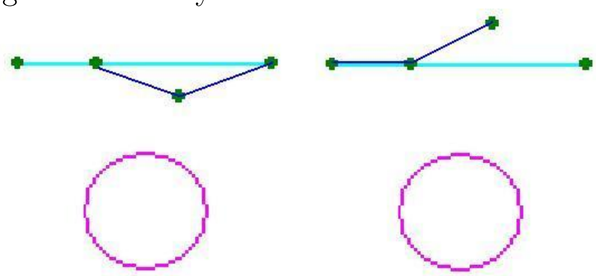
Any other symmetric, convex body B with the same chord function as A will have the endpoints of chords at $\theta = \frac{\pi}{4}n$ on its boundary. For A diamond-like with blue line segments as shown below, B has the green points on its boundary.



Lemma 2. *If three points in a row a, b, and c are on the boundary of a convex body, then the line segment connecting them is also on the boundary of the convex body.*

Proof. Assume not.

Take point q on the boundary which is not on the line segment but is on the path along the boundary from a to b to c.



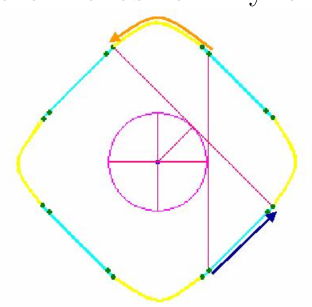
If q is inside the line segment (closer to the center), then the line between a and c is not contained in the body because q is on the boundary, so convexity is destroyed. $\Rightarrow\Leftarrow$

If q is outside the line segment, then the line segment connecting q to either a or c is not contained in the body because b is on the boundary, so convexity is destroyed. $\Rightarrow\Leftarrow$

Therefore all points on the line segment between a and c are on the boundary of the body $\square$

Theorem 1. *Among symmetric, convex bodies, a body with the property that the line passing through the left endpoint of the chord at $\theta = 0$ and the right endpoint of the chord at $\theta = \frac{3\pi}{4}$ has slope -1 or 0 is uniquely defined by its chord function.*

Proof. Let A be a body with the property described above and let B be a symmetric, convex body with the same chord function as A. By construction, the boundary of A has four line segments (segments 1, 2, 3, and 4) arranged in a diamond-like or square-like fashion. By lemmas 1 and 2, those four line segments are also on B.



The basic argument is that chords traveling along a segment generate a corner.

For the diamond-like body, traveling along the right endpoints of the chords from $\theta = 0$ to $\theta = \frac{\pi}{4}$, you are traveling along segment 4. The left endpoint of the chord at $\theta = 0$ is on segment 1 and the left endpoint of the chord at $\theta = \frac{\pi}{4}$ is on segment 2. The right endpoints of the chords from $\theta = 0$ to $\theta = \frac{\pi}{4}$ are on the boundaries of both A and B, therefore the left endpoints are also on the boundaries of both A and B. So, the boundary of B is the same as the boundary of A for the curve connecting segments 1 and 2. Similarly the curve along the boundary connecting segments 2 and 3, 3 and 4, and 4 and 1 is the same for A and B. Therefore A and B are the same.

The argument for a square-like body is the same, just rotated by $\frac{\pi}{4}$.

□

Part 2

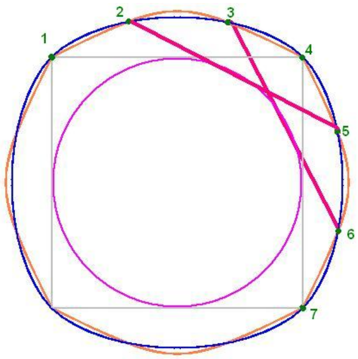
Lemma 3. *If two different bodies have the same chord function and one end of a chord is an intersection point of those two bodies, then the other end is also an intersection point.*

Proof. Assume not. Produce the two bodies and produce θ such that the chord at θ intersects both bodies at one endpoint and not the other. The length of the chord at θ is different for both bodies. $\Rightarrow\Leftarrow$ □

Theorem 2. *Among symmetric, convex bodies, any body with $c(0) = c(\frac{\pi}{2}) = c(\pi) = c(\frac{3\pi}{2}) = 2$ (around the unit circle) is uniquely defined by its chord function.*

Proof. Assume not

Take two different bodies with the same chord function and with $c(0) = c(\frac{\pi}{2}) = c(\pi) = c(\frac{3\pi}{2}) = 2$ Assume that they have a finite number of intersection points.

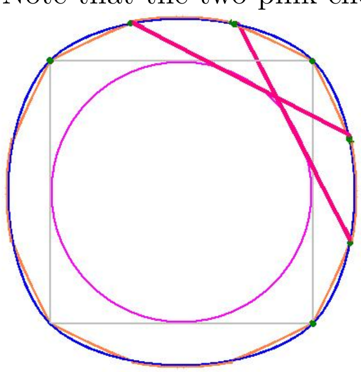


Because the bodies are symmetric, at $\theta = \frac{\pi}{2}k$ the chords have their midpoints at the place where they are tangent. Therefore, $(\pm 1, \pm 1)$ are on both bodies.

Consider points 2 and 3 such that they the closest intersections points to the y-axis and are above the x-axis. Let $n-1$ be the number of intersections between points 1 and 2. By reflection and rotation, there are $n-1$ intersections between points 3 and 4, 4 and 5, 6 and 7. The m^{th} intersection (traveling clockwise) from $(-1,1)$ should have its other endpoint at the m^{th} intersection from $(1,1)$. Assume curve orange is outside of curve blue between intersections n and $n+1$. If intersection $n+1$ is on the y-axis then orange is outside of blue between $n+1$ and $n+2$. So, orange is outside between points 2 and 3. By rotation, orange is outside between points 5 and 6 as well. Any chord of orange with an endpoint between points 5 and 6 has two endpoints outside of blue so blue's chord is shorter. $\Rightarrow\Leftarrow$

Therefore both curves are the same between points 2 and 3, 5 and 6.

Note that the two pink chords are reflections of each other along the line $y = x$.



Assume that there are only a finite number of intersection points between points 1 and 2. Take the two intersections closest to the y-axis. These will be reflections of each other along the line $y = x$. By reflection and rotation, if blue is outside in one part, then it will be outside (or the same) between the endpoints of the chords being considered. So for some θ, blue's chord at θ is longer than orange's. $\Rightarrow\Leftarrow$

Therefore both bodies have the same boundary between the endpoints of the chords being considered.

Continue this process a finite number of times to show that both bodies are the same.

$\square$