Fourier series and the Lubkin W-transform

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Abstract. We discuss the effect of a particular sequence acceleration method, the Lubkin W-transform, on the partial sums of Fourier series. We consider a very general class of functions with a single jump discontinuity, and prove that this method fails on a large set of points.

Keywords: Fourier series, Lubkin W-transform, convergence acceleration

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1. Introduction

Fourier series are a commonly used method of approximation so it is critical to analyze their speed of convergence. Many Fourier series, in particular those of functions with jump discontinuities, converge rather slowly, so it is natural to attempt to apply extrapolation methods to accelerate convergence. Various methods of acceleration have been applied with some success to the partial sums of Fourier series. In this paper, we investigate the application of one well-known method of sequence acceleration – the Lubkin W-transform– and its effect on the partial sums of Fourier series. We will show that this transform behaves poorly when applied to the Fourier series of a function with a single jump discontinuity. Some previous work of one of the authors, [1] demonstrated similar results for another well-known transform- the δ^2 process.

For a function f which is integrable on $[-\pi, \pi]$, we define the Fourier coefficients by

$$\hat{f}(n) := \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) e^{-inx} dx$$

for each integer n, and we define the nth partial sum of the Fourier series as

$$S_n f(x) := \sum_{k=-n}^{n} \hat{f}(k) e^{ikx},$$

where n is a positive integer and x is in $[-\pi, \pi]$.

When f is square-integrable, $S_n f(x)$ converges to $f(x)$ in the L^2 sense, that is, $\int_{-\pi}^{\pi} |S_n f(x) - f(x)|^2 dx \to 0$ as $n \to \infty$. Carleson [3] showed that for square-integrable f, $S_n f(x) \to f(x)$ at every point except for a set of zero Lebesgue measure. Earlier results due to Dini-Lipschitz, Lebesgue and Dirichlet-Jordan give conditions for pointwise convergence (see e.g., Zygmund [10] for these). A typical result is the Dirichlet-Jordan theorem: If f is of bounded variation over $[-\pi, \pi]$, then $S_n f(x)$ converges to $f(x)$ at each point of continuity of f.

For a numerical sequence $\{s_n\}$ which has limit s, we say a transformation s_n^* of s_n accelerates convergence if there exists a k such that each s_n^* depends only on $s_0, \ldots, s_{n+k}$ and s_n^* converges to s faster than s_n. Many sequence transformations have been developed to speed convergence of

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numerical sequences which arise in many contexts (see e.g. Brezinski [2], Delahaye [4], Sidi [7] or Wimp [9]). Consider a sequence $\{a_n\}$, $n = 1, 2, 3 \ldots$, and set $s_n = \sum_{k=0}^{n} a_n$. Let $\rho_n = a_{n+1}/a_n$. The Lubkin transform of s_n is

$$s_n^* := s_n + \frac{a_{n+1}(1-\rho_{n+1})}{1 - 2\rho_{n+1} + \rho_n\rho_{n+1}}. \tag{1}$$

This is identical to the column $\{\theta_2^j\}$ of the θ transform and the column $\{u_2^j\}$ of the Levin u-transform. When s_n is the partial sum of a geometric series, the transform (1) produces a constant sequence which has each term the sum of the geometric series. (This is true for an even wider class of sequences; e.g. see Sidi [7] or Wimp [9].)

Smith and Ford [8] used numerical tests to compare different methods of convergence acceleration on the partial sums of Fourier series. Using a set of five points they tested slowly and rapidly converging Fourier series. Among the methods they tested were the θ and u tranforms, which showed some improvement of convergence in some cases. Drummond [5] discusses many methods of convergence acceleration and includes discussion of their application to Fourier series.

Although with specific series it is possible to get better approximations – even dramatically better results at some points – the Lubkin-W transformation does not in general behave well. We will show that for a fairly general set of functions, and for a large set of values $x \in [-\pi, \pi]$, this transform turns the sequence $\{S_n f(x) : n = 1, 2, \ldots\}$ into a sequence without a limit.

Theorem 1. *Suppose that $f \in C^2([-\pi, \pi])$ and that $f(-\pi) \neq f(\pi)$. Consider the sequence $S_n^* f(x)$ formed by applying the transformation (1) to the sequence $S_n f(x)$. Then $S^* f_n(x)$ diverges at every x of the form $x = 2\pi a$, where $a \in (-\frac{1}{4}, \frac{1}{4})$ is irrational.*

Remarks. Consider the 2π periodic extension of f to the whole real line. The hypotheses of the theorem implies that this function is in C^2, except for a single jump discontinuity every period. By periodicity, the Theorem remains valid if this jump occurs anywhere in the interval $[-\pi, \pi]$.

At a jump discontinuity of f, the partial sums of the Fourier series exhibit the Gibbs phenomenon in a neighborhood of the discontinuity, and many authors report difficulties with acceleration methods near such a discontinuity. Our results show that if one applies the Lubkin transform to the partial sums of a Fourier series with a jump discontinuity, then curiously, difficulties appear away from the discontinuity. The theorem should be compared to those found in [1] where similar results are shown for the δ^2 process. Our proof will follow a few ideas found in that paper, although here the expressions involved in the transforms are more complicated and consequently the analysis is more difficult.

2. The proof of Theorem 1

If $S_n f(x)$ denotes the sequence of partial sums of the Fourier series of f, applying the Lubkin transform (1) results in the sequence of functions:

$$S_n^* f(x) = S_n f(x) + \frac{a_{n+1}(1-\rho_{n+1})}{1 - 2\rho_{n+1} + \rho_n\rho_{n+1}} \tag{2}$$

where

$$a_n = S_n f(x) - S_{n-1} f(x) = \hat{f}(-n)e^{-inx} + \hat{f}(n)e^{inx} \tag{3}$$

and

$$\rho_n = \frac{a_{n+1}}{a_n} = \frac{\hat{f}(-(n+1))e^{-i(n+1)x} + f(n+1)e^{i(n+1)x}}{\hat{f}(-n)e^{-inx} + \hat{f}(n)e^{inx}} \tag{4}$$

Since the functions we are considering have bounded variation, the Dirichlet-Jordan theorem applies so that $S_n f(x)$ converges to $f(x)$ at every point of the interval $(-\pi,\pi)$. Therefore, we will show that the fraction $\frac{a_{n+1}(1-\rho_{n+1})}{1-2\rho_{n+1}+\rho_n\rho_{n+1}}$ has a subsequence which stays away from 0 at all x of the form $x = 2\pi a$, $a \in (-\frac{1}{4},\frac{1}{4})$, a irrational.

Lemma 1. *Let $f \in C^2([-\pi,\pi])$. Then for every integer n,*

$$\hat{f}(n) = \frac{(-1)^{n+1}}{2in}\alpha + \frac{(-1)^n}{2n^2}\beta - \frac{\widehat{f''}(n)}{n^2},$$

where $\alpha = [f(\pi) - f(-\pi)]/\pi$ and $\beta = [f'(\pi) - f'(-\pi)]/\pi$.

This is easily shown using integration by parts. Notice also that $\frac{\widehat{f''}(n)}{n^2} = o(\frac{1}{n^2})$ as $|n| \to \infty$ by the Riemann-Lebesgue lemma. (See, e.g. Zygmund [10], pg. 45.)

Using the result of Lemma 1 in equation (3) gives

$$\begin{aligned} a_n &= \hat{f}(-n)e^{-inx} + \hat{f}(n)e^{inx} \\ &= \left(\frac{(-1)^{n+1}}{2in}\alpha + \frac{(-1)^n}{2n^2}\beta - \frac{\widehat{f''}(n)}{n^2}\right)e^{inx} + \left(-\frac{(-1)^{-(n+1)}}{2in}\alpha + \frac{(-1)^{-n}}{2n^2}\beta - \frac{\widehat{f''}(-n)}{n^2}\right)e^{-inx} \\ &= \frac{\alpha(-1)^{n+1}}{n}\sin(nx) + \frac{\beta(-1)^n}{n^2}\cos(nx) - \frac{\widehat{f''}(-n)}{n^2}e^{-inx} - \frac{\widehat{f''}(n)}{n^2}e^{inx} \end{aligned} \tag{5}$$

Now substitute (5) into (4) and (2) and simplify to see that the fraction $\frac{a_{n+1}(1-\rho_{n+1})}{1-2\rho_{n+1}+\rho_n\rho_{n+1}}$ in (2) behaves asymtotically as the following

$$\frac{\alpha(-1)^{n+2}\left(\frac{\sin\left((n+1)x\right)}{n+1}\right)\left(1 - \frac{(-1)^{n+3}(n+1)\sin\left((n+2)x\right)}{(-1)^{n+2}(n+2)\sin\left((n+1)x\right)}\right)}{1 - 2\left(\frac{(-1)^{n+3}(n+1)\sin\left((n+2)x\right)}{(-1)^{n+2}(n+2)\sin\left((n+1)x\right)}\right) + \left(\frac{(-1)^{n+2}n\sin\left((n+1)x\right)}{(-1)^{n+1}(n+1)\sin(nx)}\right)\left(\frac{(-1)^{n+3}(n+1)\sin\left((n+2)x\right)}{(-1)^{n+2}(n+2)\sin\left((n+1)x\right)}\right)}$$

which simplifies to

$$\frac{\alpha(-1)^{n+2}\sin(nx)\sin\left((n+1)x\right)\left(\frac{\sin\left((n+1)x\right)}{n+1} + \frac{\sin\left((n+2)x\right)}{n+2}\right)}{\sin(nx)\sin\left((n+1)x\right) + 2\frac{n+1}{n+2}\sin(nx)\sin\left((n+2)x\right) + \frac{n}{n+2}\sin\left((n+1)x\right)\sin\left((n+2)x\right)}$$

It will be computationally easier to work with this expression if we shift the indices back by one. Thus, we consider the expression

$$\frac{\alpha(-1)^{n+1}\sin\left((n-1)x\right)\sin(nx)\left(\frac{\sin(nx)}{n} + \frac{\sin\left((n+1)x\right)}{n+1}\right)}{\sin\left((n-1)x\right)\sin(nx) + 2\frac{n}{n+1}\sin\left((n-1)x\right)\sin\left((n+1)x\right) + \frac{n-1}{n+1}\sin(nx)\sin\left((n+1)x\right)}. \tag{6}$$

We complete the proof of the theorem using two lemmas: the first shows that if x satisfies the hypotheses of the theorem, then there exists a subsequence $\{n_k\}$ such that the denominator of (6) is bounded by $\frac{4+48\pi}{n_k}$. The second demonstrates that for this subsequence, the numerator of (6) is bounded below by $\frac{c}{n_k}$ for some constant c. Thus, along this subsequence, (6) cannot converge to 0, thus causing the sequence $S_n^* f(x)$ in equation (2) to not converge to $f(x)$.

Lemma 2. *Given $x = 2\pi a$, $a \in (-\frac{1}{4}, \frac{1}{4})$, a irrational, there exists an infinite number of n_k such that*

$$\left|\sin\left((n_k-1)x\right)\sin n_k x + 2\frac{n_k}{n_k+1}\sin\left((n_k-1)x\right)\sin\left((n_k+1)x\right) + \frac{n_k-1}{n_k+1}\sin n_k x \sin\left((n_k+1)x\right)\right| < \frac{4+48\pi}{n_k} \tag{7}$$

Proof. Notice that since

$$\Big|\sin\left((n-1)x\right)\sin(nx) + 2\frac{n}{n+1}\sin\left((n+1)x\right)\sin\left((n-1)x\right) + \frac{n-1}{n+1}\sin(nx)\sin\left((n+1)x\right) - \Big(\sin\left((n-1)x\right)\sin(nx) + 2\sin\left((n+1)x\right)\sin\left((n-1)x\right) + \sin(nx)\sin\left((n+1)x\right)\Big)\Big| < \frac{4}{n} \tag{8}$$

it suffices to show that there exists a sequence $\{n_k\}$ such that

$$\Big|\sin\left((n_k-1)x\right)\sin(n_k x) + 2\sin\left((n_k-1)x\right)\sin\left((n_k+1)x\right) + \sin(n_k x)\sin\left((n_k+1)x\right)\Big| < \frac{48\pi}{n_k} \tag{9}$$

Employing trigonometric identities on the terms inside the absolute value, we obtain

$$\begin{aligned}&\sin(n_k x)\Big(\sin(n_k x)\cos(x) - \cos(n_k x)\sin(x)\Big) + 2\Big(\sin(n_k x)\cos(x) - \cos(n_k x)\sin(x)\Big)\times\\ &\Big(\sin(n_k x)\cos(x) + \cos(n_k x)\sin(x)\Big) + \sin(n_k x)\Big(\sin(n_k x)\cos(x) + \cos(n_k x)\sin(x)\Big)\\ &= 2\sin^2(n_k x)\cos(x) + 2\sin^2(n_k x)\cos^2(x) - 2\cos^2(n_k x)\sin^2(x)\end{aligned} \tag{10}$$

Now consider the equation $\sin^2(\beta)\cos(x) + \sin^2(\beta)\cos^2(x) - \cos^2(\beta)\sin^2(x) = 0$. Rearranging we have

$$\tan^2(\beta) = \frac{\sin^2(x)}{\cos(x)(1+\cos(x))} \tag{11}$$

The square of the tangent function has range $[0, +\infty)$. Given any $x \in (\frac{-\pi}{2}, \frac{\pi}{2})$ there exists a $\beta \in (0, \frac{\pi}{2})$ such that equation (11) is satisfied.

Fix x such that x has the form $x = 2\pi a$, $a \in (-\frac{1}{4}, \frac{1}{4})$, a is irrational, and let β be given by (11). Consider the function $g(y) = \sin(y-x)\,\sin(y) + 2\sin(y-x)\,\sin(y+x) + \sin(y)\,\sin(y+x)$. Then $g(\beta) = 0$ and $|g'(y)| \le 8$ for all y. We claim that there exists a sequence n_k (depending on x) such that $|(n_k x - \beta) \pmod{2\pi}| \le \frac{6\pi}{n_k}$. Assume this claim momentarily. Then for these n_k, $|g(n_k x)| = |g(n_k x) - g(\beta)| \le 8|(n_k x - \beta) \pmod{2\pi}| \le \frac{48\pi}{n_k}$. This shows (9) and combined with (8) gives (7).

We need to substantiate the claim. This follows from a result due to Chebyshev (see A. Ya. Kinchin [6], p.39): For an arbitrary irrational number a and arbitrary real number γ, there exists an infinite sequence of positive integers $\{n_k\}$ such that $|n_k a - m - \gamma| \le \frac{3}{n_k}$ for some integer m (which depends on n_k.) Apply this with a as above and $\gamma = \frac{\beta}{2\pi}$ and multiply the resulting equation by 2π to obtain $|n_k x - 2\pi m - \beta| \le \frac{6\pi}{n_k}$. This is the claim, which finishes the proof of the Lemma.

We must now examine the numerator of (6).

Lemma 3. *For x and $\{n_k\}$ as in Lemma 2, the numerator*

$$\alpha(-1)^{n_k+1}\sin\left((n_k-1)x\right)\sin(n_k x)\left(\frac{\sin(n_k x)}{n_k} + \frac{\sin\left((n_k+1)x\right)}{n_k+1}\right)$$

is bounded below, for n_k large, by $\frac{c}{n_k}$ for some constant c (which depends on x but not n_k).

Proof. Since

$$\left|\alpha(-1)^{n_k+1}\sin(n_kx)\sin\left((n_k-1)x\right)\left(\frac{\sin(n_kx)}{n_k}+\frac{\sin\left((n_k+1)x\right)}{n_k+1}\right)\right.$$
$$\left.-\left(\alpha(-1)^{n_k+1}\sin(n_kx)\sin\left((n_k-1)x\right)\left(\frac{\sin(n_kx)}{n_k}+\frac{\sin\left((n_k+1)x\right)}{n_k}\right)\right)\right|<\frac{\alpha}{n_k^2}$$

it suffices to show that

$$\left|\alpha(-1)^{n+1}\sin(n_kx)\sin\left((n_k-1)x\right)\left(\frac{\sin(n_kx)}{n_k}+\frac{\sin\left((n_k+1)x\right)}{n_k}\right)\right|>\frac{c}{n_k}$$

that is,

$$\left|\alpha(-1)^{n+1}\sin(n_kx)\sin\left((n_k-1)x\right)\Big(\sin(n_kx)+\sin\left((n_k+1)x\right)\Big)\right|>c. \tag{12}$$

Again, using trigonometric identities, we rewrite the trigonometric terms in (12) as

$$\sin(n_kx)\Big(\sin(n_kx)\cos(x)-\cos(n_kx)\sin(x)\Big)\Big(\sin(n_kx)+\sin(n_kx)\cos(x)+\cos(n_kx)\sin(x)\Big)$$
$$=\sin(n_kx)\Big(\sin^2(n_kx)\cos(x)+\sin^2(n_kx)\cos^2(x)-\cos^2(n_kx)\sin^2(x)\Big)-\sin^2(n_kx)\cos(n_kx)\sin(x)$$

Notice that the terms in the large parentheses match equation (10). As in the proof of Lemma 1, these terms are bounded above by $\frac{48\pi}{n_k}$. Consider the term $-\sin^2(n_kx)\cos(n_kx)\sin(x)$. By continuity, since $n_kx\to\beta$, (mod 2π), then $-\sin^2(n_kx)\cos(n_kx)\sin(x)\to-\sin^2(\beta)\cos(\beta)\sin(x)$. Since $x\neq 0$, $\sin(x)\neq 0$, and since $\beta\in(0,\frac{\pi}{2})$, $\sin^2(\beta)\neq 0$ and $\cos(\beta)\neq 0$. Thus, $-\sin^2(n_kx)\cos(n_kx)\sin(x)\to -\sin^2(\beta)\cos(\beta)\sin(x)\neq 0$ and this allows us to conclude (12).

3. Remarks and further results

The theorem states that for $x=2\pi a$, $a\in(-\frac{1}{4},\frac{1}{4})$ $S_n^*f(x)$ does not converge to $f(x)$. It is quite likely that this convergence fails at other x also but we have not been able to show this. The methods used in the proof apply in other situations. For example, consider the slowly converging series $\sum_{j=0}^{\infty}\frac{\sin jx}{j}$. With only slight alterations, left to the reader, the proof of Theorem 1 can be adapted to show:

Theorem 2. *Set $S_n(x)=\sum_{j=0}^{\infty}\frac{\sin jx}{j}$ and let $S_n^*(x)$ be the sequence of functions obtained by applying (1) to $S_n(x)$. Then $S_n^*(x)$ fails to converge at any point $x=2\pi a$ where $a\in(\frac{1}{4},\frac{3}{4})$ is irrational.*

Consider the function $f(x)=x$. Figure 1 shows the partial sums of its Fourier series, $S_nf(x)$, for $n=25$ and 100 on the interval $[0,\pi]$. (f as well as all S_nf are odd functions.) The transformed partial sums $S_n^*f(x)$ are shown for the same values of n in Figure 2. The graphs show the difficulties which occur at a dense set of values in $(\frac{-\pi}{2},\frac{\pi}{2})$. Figure 3 investigates whether the transformed sequence provides a better approximation by graphing $|f(x)-S_nf(x)|-|f(x)-S_n^*f(x)|$. At points where this is positive, $S_n^*f(x)$ is the better approximation; where this is negative, $S_nf(x)$ provides the better approximation. It would be interesting to be able to *a priori* determine at which x the transform speeds convergence, but this seems difficult.

It is likely that the methods in this paper could be applied to show that other sequence acceleration transforms may give unreliable results when applied to Fourier series. We suspect that similar difficulties can be proven to exist for other transforms. Currently, each transform considered seems to require a different analysis so it would also be of interest to develop more general methods for these type of theorems. These are topics for future study.

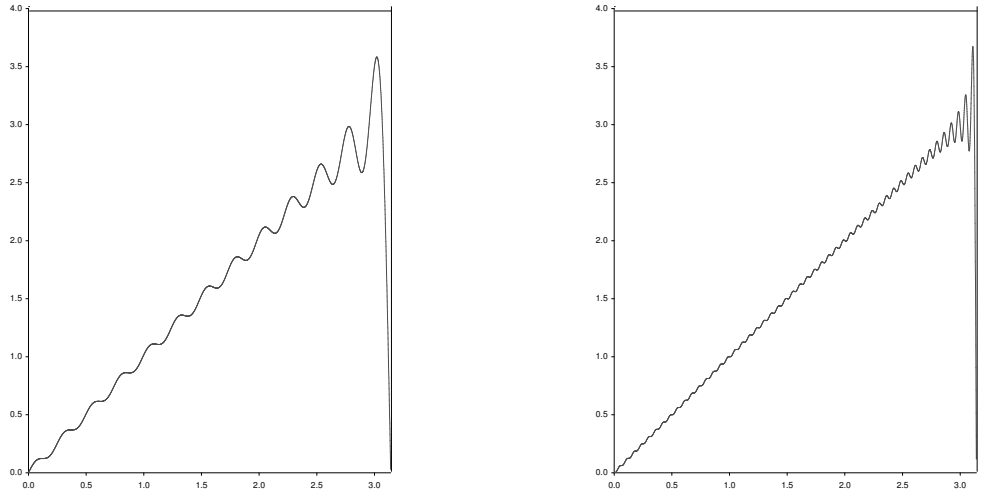

Figure 1. Partial sums for $n = 25$ and $n = 100$

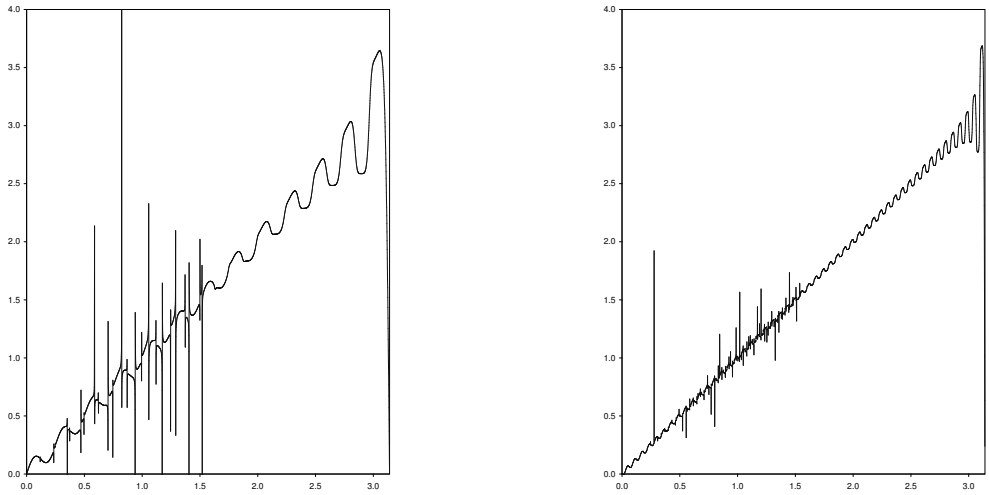

Figure 2. Transformed partial sums for $n = 25$ and $n = 100$

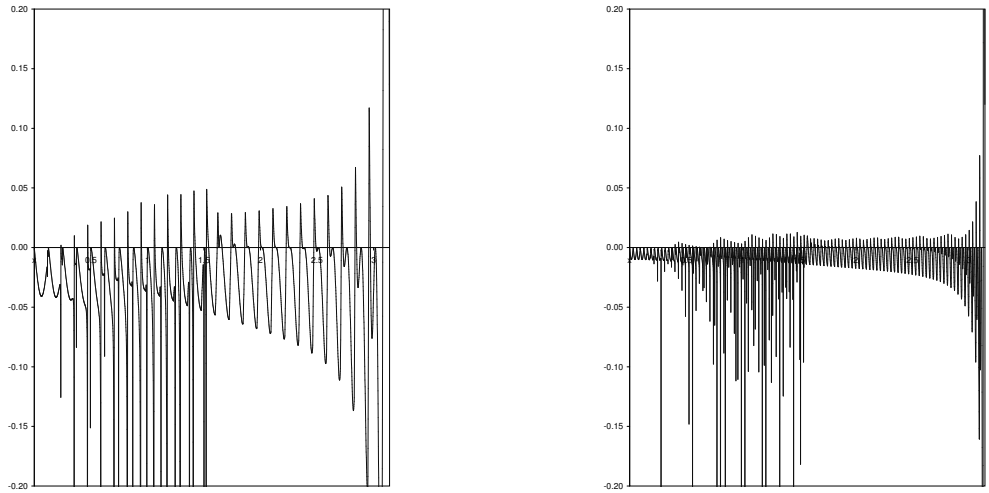

Figure 3. $|f(x) - S_n f(x)| - |f(x) - S_n^* f(x)|$ for $n = 25$ and $n = 100$

References

1. Abebe, E., Graber, J., and Moore, C. N.: Fourier series and the δ^2 process, preprint
2. Brezinski, C.: Accélération de la Convergence en Analyse Numŕique, Springer Verlag, Berlin Heidelberg New York (1977)
3. Carleson, L.: On convergence and growth of partial sums of Fourier series. Acta. Math. **116**, 135-157 (1966)
4. Delahaye, J.-P.: Sequence Transformations, Springer Series in Computational Mathematics 11, Springer Verlag, Berlin, (1988)
5. Drummond, J.E.: Convergence speeding, convergence and summability, Journal of Computational and Applied Mathematics **11**, 145-159 (1984)
6. Kinchin, A. Ya.: Continued Fractions, The University of Chicago Press (1964)
7. Sidi, A.: Practical Extrapolation Methods, Cambridge Monographs on Applied and Computational Mathematics 10, Cambridge University Press (2003)
8. Smith, D. A. and Ford, W. F.: Numerical Comparisons of Nonlinear Convergence Accelerators, Mathematics of Computation **38**, 481-499 (1982)
9. Wimp, J.: Sequence Transformations and Their Applications, Academic Press, New York (1981)
10. Zygmund, A.: Trigonometric Series, Second Edition, Cambridge University Press, Cambridge (1959)