Floating Drops in $\mathbb{R}^2$

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July 27, 2007

Abstract

Using methods developed for three dimensional drops, we study a two dimensional drop in a $\mathbb{R}^2$. The differential equations describing the curves that make up these drops can be found using known derivations of the surface curvature. Solving this problem numerically and explicitly with parameters for densities, surface tensions, and the prescribed radius of the triple junction results in pictorial representations of the drop.

1 Introduction

Three different liquids make up a drop: the top liquid, usually air, which we will call liquid (0); the liquid contained within the drop, which we will call liquid (1); and the most dense liquid supporting the drop, which we will call liquid (2). The three different liquids must each have a different density, $\rho_0 < \rho_1 < \rho_2$, so that ρ_0, ρ_1, and ρ_2 correspond to liquids (0), (1), and (2) respectively. Between each liquid there must also be surface tensions: σ_{01}, σ_{02}, and σ_{12}, corresponding to the tension between liquids (0) and (1), (0) and (2), and (1) and (2) respectively. Each surface tension must be greater than zero or else mixing would occur. Also, since the drop is in equilibrium, the sum of all forces in each direction must be zero, so each surface tension must be less than or equal to the vector sum of the other two. We will need capillary constants as defined by Elcrat, Neel and Siegel [2]:

$$\begin{aligned} \kappa_{01} &= \frac{(\rho_1 - \rho_0)g}{\sigma_{01}} \\ \kappa_{02} &= \frac{(\rho_2 - \rho_0)g}{\sigma_{02}} \\ \kappa_{12} &= \frac{(\rho_2 - \rho_1)g}{\sigma_{12}} \end{aligned} \tag{1}$$

We will call the top curve of the drop u, the bottom curve v and the outer curve w. The angles between each curve are γ_{01} (between v and w), γ_{02} (between u and v), and γ_{12} (between u and w) (see Figure 1).

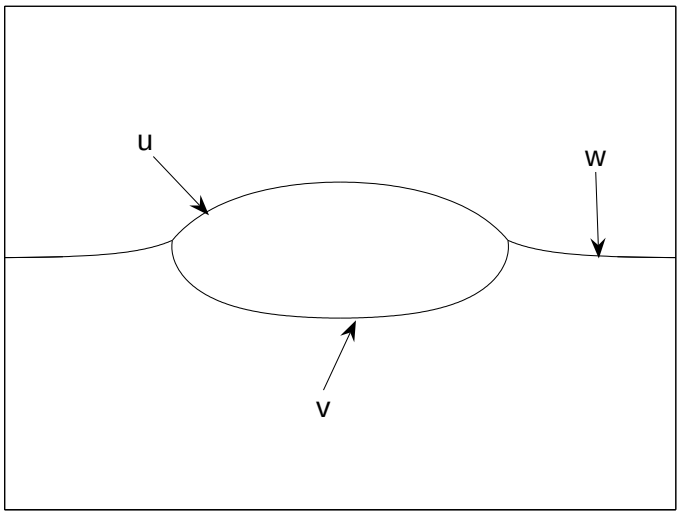

Figure 1: Drop Surfaces

Adopting what was derived in [2], the following equations hold:

$$\begin{aligned} K(u) &= \kappa_{01} u + \frac{\lambda}{\sigma_{01}} \\ K(v) &= \kappa_{12} v - \frac{\lambda}{\sigma_{12}} \\ K(w) &= \kappa_{02} w \end{aligned} \tag{2}$$

Where λ is the Lagrange multiplier which is determined as the problem is solved and K is the curvature function as used by Wente [5]. In order to find the physical curves u and v

$$u = -U - \frac{\lambda}{\kappa_{01}\,\sigma_{01}} \tag{3}$$

and

$$v = V + \frac{\lambda}{\kappa_{12}\,\sigma_{12}}. \tag{4}$$

Where

$$\begin{aligned} K(U) &= \kappa U \\ K(V) &= \kappa V \end{aligned} \tag{5}$$

Now,

$$K(u) = \frac{d}{dx} \sin \psi$$

Where ψ is the inclination angle, $x(\psi)$ is the x-coordinate and $u(\psi)$, $v(\psi)$ and $w(\psi)$ are the z-coordinates of the various curves.

Now we have an equation that can be solved for any one of the three curves.

$$\frac{d}{dx}\sin\psi = \kappa u \tag{6}$$

With these basic conditions and equations, we study (6) as a building block for the floating drop problem.

2 Derivations

From $\frac{d}{dx}\sin\psi = \kappa u$, we can derive two second order differential equations for the sessile curves parameterized by inclination angle. Differentiating with respect to ψ, we get the equation $\cos\psi\,\frac{d\psi}{dx} = \kappa u$ or

$$\frac{dx}{d\psi} = \frac{\cos\psi}{\kappa u}. \tag{7}$$

Noting that $\frac{du}{dx} = \tan\psi$, we also see that

$$\frac{du}{d\psi} = \frac{\sin\psi}{\kappa u}. \tag{8}$$

We integrate (8) to obtain a value for $u(\psi)$

$$\begin{aligned}
\int_{u_0}^{u} \kappa u\, du &= \int_0^{\psi} \sin\psi\, d\psi \\
\frac{\kappa}{2}(u^2 - u_0^2) &= 1 - \cos\psi \\
u(\psi) &= \pm\sqrt{u_0^2 + \frac{2}{\kappa}(1-\cos\psi)}.
\end{aligned} \tag{9}$$

We will use the positive square root, because we are dealing with curves above the x-axis. With this equation, we can now integrate equation (7) to find $x(\psi)$.

$$\begin{aligned}
\int_{x_0}^{x} dx &= \int_0^{\psi} \frac{\cos\psi'}{\kappa\sqrt{u_0^2 + \frac{2}{\kappa}(1-\cos\psi')}}\, d\psi' \\
x(\psi) &= \int_0^{\psi} \frac{\cos\psi'}{\kappa\sqrt{u_0^2 + \frac{2}{\kappa}(1-\cos\psi')}}\, d\psi' + const.
\end{aligned} \tag{10}$$

We now have equations for both $u(\psi)$ and $x(\psi)$.

If $u_0 \neq 0$, this integral may be simplified using elliptical integrals, however, if $u_0 = 0$, we are able to explicitly integrate as in Bhatnagar and Finn [1].

$$\begin{aligned}
x(\psi) &= \int_0^{\psi} \frac{\cos\psi'}{\sqrt{2\kappa(1-\cos\psi')}}\, d\psi' + const. \\
x(\psi) &= \frac{1}{\sqrt{2\kappa}}\int_0^{\psi} \frac{\cos\psi'}{\sqrt{1-\cos\psi'}}\, d\psi' + const.
\end{aligned} \tag{11}$$

If we substitute $u = \cos\psi$, we then have $\psi = \cos^{-1} u$ and $d\psi = \frac{-1}{\sqrt{1-u^2}}\, du$.

$$x(\psi) = \frac{1}{\sqrt{2\kappa}} \int_1^{\cos\psi} \frac{u}{\sqrt{1-u}} \frac{-1}{\sqrt{1-u^2}}\, du + const.$$

Integrating by parts:

$$\begin{aligned}
\frac{1}{\sqrt{2\kappa}} \int_1^{\cos\psi} \frac{-1}{\sqrt{1-u}} \frac{u}{\sqrt{1-u^2}}\, du &= \frac{\sqrt{1-u^2}}{\sqrt{1-u}} - \int_1^{\cos\psi} \left(\sqrt{1-u^2}\right) \left(\frac{(1-u)^{-3/2}}{2}\right) du \\
&= \sqrt{1+u} - \int_1^{\cos\psi} \frac{\sqrt{1-u}\sqrt{1+u}}{2(1-u)\sqrt{1-u}}\, du \\
&= \sqrt{1+u} - \int_1^{\cos\psi} \frac{\sqrt{1+u}}{2(1-u)}\, du
\end{aligned}$$

Substituting $v = \sqrt{1+u}$

$$\begin{aligned}
\int_1^{\cos\psi} \frac{\sqrt{1+u}}{2(1-u)}\, du &= \int_{\sqrt{2}}^{\sqrt{1+\cos\psi}} \frac{v^2}{2-v^2}\, dv \\
&= \int_{\sqrt{2}}^{\sqrt{1+\cos\psi}} -1 + \frac{2}{2-v^2}\, dv \\
&= -\sqrt{1+\cos\psi} + 2\int_{\sqrt{2}}^{\sqrt{1+\cos\psi}} \frac{1}{(\sqrt{2}-v)(\sqrt{2}+v)}\, dv + const.
\end{aligned}$$

Partial fractions:

$$\begin{aligned}
\int_1^{\cos\psi} \frac{\sqrt{1+u}}{2(1-u)} &= -\sqrt{1+\cos\psi} + 2\int_{\sqrt{2}}^{\sqrt{1+\cos\psi}} \frac{1}{4}\frac{\sqrt{2}}{(\sqrt{2}-v)} + \frac{1}{4}\frac{\sqrt{2}}{(\sqrt{2}+v)}\, dv + const. \\
&= -\sqrt{1+\cos\psi} + \frac{\sqrt{2}}{2}\int_{\sqrt{2}}^{\sqrt{1+\cos\psi}} \frac{1}{\sqrt{2}-v}\, dv + \frac{\sqrt{2}}{2}\int_{\sqrt{2}}^{\sqrt{1+\cos\psi}} \frac{1}{\sqrt{2}+v}\, dv + const. \\
&= -\sqrt{1+\cos\psi} - \frac{\sqrt{2}}{2}\ln|\sqrt{2}-\sqrt{1+\cos\psi}| + \frac{\sqrt{2}}{2}\ln|\sqrt{2}+\sqrt{1+\cos\psi}| + const. \\
&= -\sqrt{1+\cos\psi} + \frac{\sqrt{2}}{2}\ln\left[\frac{|\sqrt{2}+\sqrt{1+\cos\psi}|}{|\sqrt{2}-\sqrt{1+\cos\psi}}\right] + const. \\
&= -\sqrt{1+\cos\psi} + \frac{\sqrt{2}}{2}\ln\left[\frac{\sqrt{2}}{\sqrt{2}}\frac{|1+\left(\frac{\sqrt{1+\cos\psi}}{\sqrt{2}}\right)|}{|1-\left(\frac{\sqrt{1+\cos\psi}}{\sqrt{2}}\right)}\right] + const.
\end{aligned}$$

Now $\tanh^{-1}\sqrt{\frac{1+\cos\psi}{2}} = \frac{1}{2}\ln\frac{1+\sqrt{\frac{1+\cos\psi}{2}}}{1-\sqrt{\frac{1+\cos\psi}{2}}}$,

$$\int_1^{\cos\psi} \frac{\sqrt{1+u}}{2(1-u)} = -\sqrt{1+\cos\psi} + \sqrt{2}\tanh^{-1}\left(\sqrt{\frac{1+\cos\psi}{2}}\right) + const.$$

So now:

$$
\begin{aligned}
x(\psi) &= \frac{1}{\sqrt{2\kappa}}\left[2\sqrt{1+\cos\psi} - \sqrt{2}\tanh^{-1}\left(\sqrt{\frac{1+\cos\psi}{2}}\right)\right] + const. \\
x(\psi) &= \frac{1}{\sqrt{\kappa}}\left[\sqrt{2(1+\cos\psi)} - \tanh^{-1}\left(\sqrt{\frac{1+\cos\psi}{2}}\right)\right] + const. \qquad (12)
\end{aligned}
$$

Now we have the integrated forms of both $u(\psi)$ and $x(\psi)$, which may be reflected to create all curves of the drop.

3 Theorem

Theorem 3.1. *For any initial values $u_0 > 0$, $x_0 = 0$, the solution of*

$$\frac{du}{d\psi} = \frac{\sin\psi}{\kappa u}, \quad \frac{dx}{d\psi} = \frac{\cos\psi}{\kappa u}, \qquad (13)$$

can be found for the range $0 < \psi \leqslant \pi$ with a monotonically increasing u. The function $x(\psi)$ reaches its maximum value when $\psi = \frac{\pi}{2}$, and then decreases again to a value $\bar{x} = \lim_{\psi\to\pi} x(\psi) > 0$.

Proof. Since the derivative of $x(\psi)$ is known, the maximum value for the function can be found by setting that derivative equal to zero. Doing so, it can be seen that on the interval $0 < \psi \leqslant \pi$, a maximum occurs only once, when $\psi = \frac{\pi}{2}$. For the region where $\psi > \frac{\pi}{2}$, $\frac{dx}{d\psi} < 0$, which mean that the curve must have a "fold over" section, or two u values for every x value on part of the curve. Following Finn [4], the equation, $\frac{d}{dx}\sin\psi = \kappa u$ can be integrated on both sides with respect to x. Denote the values for the upper curve $\sin\psi^+$ and u^+ and the values for the lower curve $\sin\psi^-$ and u^-, then subtracting gives

$$(\sin\psi^- - \sin\psi^+) = \kappa\int_x^{x_{max}} (u^+ - u^-)dx'. \qquad (14)$$

From this equation, we can see that, near x_{max}, $u^+ > u^-$, thus $\sin\psi^- > \sin\psi^+$. From (13), u^+ and u^- are increasing in ψ so the inequalities, $u^+ > u^-$ and $\sin\psi^- > \sin\psi^+$, continue to hold for all ψ. If we set $\psi^+ = \pi$, then we cannot have $\bar{x} \leqslant 0$, as equation (14) would result in $0 < 0$ Therefore, a limit $\bar{x} = \lim_{\psi\to\pi} x(\psi) > 0$ must exist.

□

Conjecture 3.2. *The boundary value problem*

$$\frac{du}{d\psi} = \frac{\sin\psi}{\kappa u}, \qquad \frac{dx}{d\psi} = \frac{\cos\psi}{\kappa u},$$

$$x(0) = 0 \qquad x(\bar{\psi}) = \bar{x}$$

has a unique solution for all $\bar{x} > 0$, $\bar{\psi} \in (0, \pi]$.

A method for proving this conjecture would be to show that as $u_0 \to \infty$, $x(\psi) \to 0$, and as $u_0 \to 0$, $x(\psi) \to \infty$ for all $\psi \in (0, \pi]$. Then by continuous dependence on the data u_0, there would be $u_0 > 0$ so that the solution to the initial value problem with $x(0) = 0$, $u(0) = u_0$ would also solve the boundary value problem. Some progress has occurred with this scheme.

Since

$$u(\psi) = \sqrt{\frac{2}{\kappa}(1 - \cos\psi) + u_0^2}$$

we can see that the term $\frac{2}{\kappa}(1 - \cos\psi)$ is never greater than $\frac{4}{\kappa}$. Therefore as $u_0 \to \infty$, u_0^2 will dominate the equation. Since $u \geq u_0$, $u(\psi) \to \infty$ as well, for $\psi \in [0, \pi]$. With this fact in mind, the equation

$$x(\psi) \leq \int_0^{\bar{\psi}} \frac{\cos\psi}{\kappa u_0} = \frac{1}{\kappa u_0} \int_0^{\bar{\psi}} \cos\psi = \frac{\sin\bar{\psi}}{\kappa u_0}$$

shows that $x(\psi) \to 0$ as $u_0 \to \infty$ again for $\psi \in [0, \pi]$.

4 Numerics

We find numerically the drop curves given the densities and surface tensions of the three liquids. The numerics used were modeled after Elcrat and Treinen's [3] work on the three dimensional floating drop. The inner curves, u and v used the initial differential equations

$$\frac{du}{d\psi} = \frac{\sin\psi}{\kappa u}, \quad \frac{dx}{d\psi} = \frac{\cos\psi}{\kappa u},$$

and the *ode45* function to call and plot the upper and lower curves of the drop (U and V). Each curve used the *fzero* function to find the u_0 so that the curve ended with the desired $\bar{\psi}$ and $\bar{x}$. The outer curve used the formula

$$x(\psi) = \frac{1}{\sqrt{\kappa}} \left[\sqrt{2(1 + \cos\psi)} - \tanh^{-1}\left(\sqrt{\frac{1 + \cos\psi}{2}} \right) \right] + const.,$$

We match the physical curves $u(\bar{\psi})$ and $v(\bar{\psi})$ by the relations

$$\begin{aligned} -\bar{U} - \frac{\lambda}{\kappa_{01}\sigma_{01}} &= \bar{V} + \frac{\lambda}{\kappa_{12}\sigma_{12}} \\ -\left[\bar{U} + \bar{V}\right] &= \lambda \left[\frac{1}{\kappa_{01}\sigma_{01}} + \frac{1}{\kappa_{12}\sigma_{12}} \right]. \end{aligned}$$

Then we have the difference between $v(\bar{\psi})$ and $w(\bar{\psi})$

$$F(\psi) = \frac{\kappa_{12}\sigma_{12}\bar{V} - \kappa_{01}\sigma_{12}\bar{U}}{\kappa_{01}\sigma_{01} + \kappa_{12}\sigma_{12}} - \bar{w}. \tag{15}$$

We want this equation to be equal to zero to make all three curves match. The curves $\bar{U}$ and $\bar{V}$ are greater than or equal to zero for all values of ψ. We know that if $\psi = 0$, $\bar{V} \equiv 0$, $\bar{U} > 0$ and $\bar{w} > 0$. Therefore, for these parameters, $F(0) < 0$. We also know that if $\psi = \gamma_{02}$,

$\bar{U} \equiv 0$, $\bar{V} > 0$ and $\bar{w} < 0$. For this variation of parameters, $F(\gamma_{02}) > 0$, so by the Intermediate Value Theorem, $F(\bar{\psi}) = 0$ for some value $0 < \bar{\psi} < \gamma_{02}$ from [2]. When $F(0) = 0$, the three curves u, v, and w all match up and form a drop. We find this zero value for the function $F(\psi)$ using the *fzero* function as modeled by Elcrat and Treinen [3].

Figures were created describing different drop scenarios (see Figs. 2-5). In Figure 2, the densities of the drop and the liquid below it are very similar making a dense drop and Figure 3 has a larger radius. In Figure 4, the surface tension of the top curve is equal to the sum of the other two surface tensions, providing tangential curves on the top of the drop. We can see in Figure 5, for small drops (the radius here is 0.1), the drop appears almost circular, and becomes more so as the radius is decreased. This correlates to the same case in three dimensions.

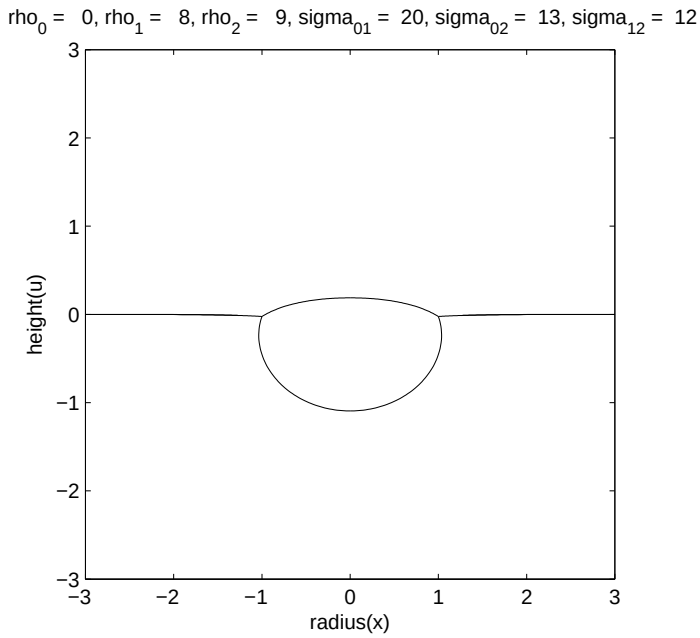

Figure 2: Dense Drop

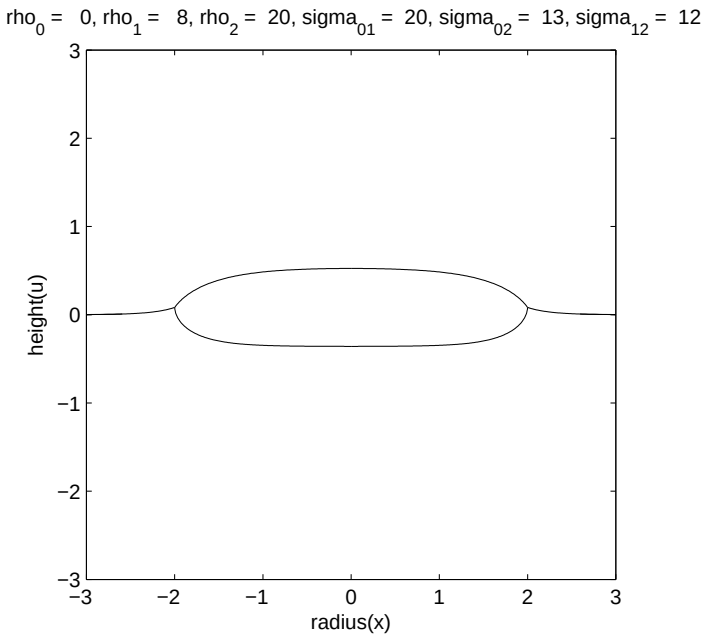

Figure 3: Large Drop

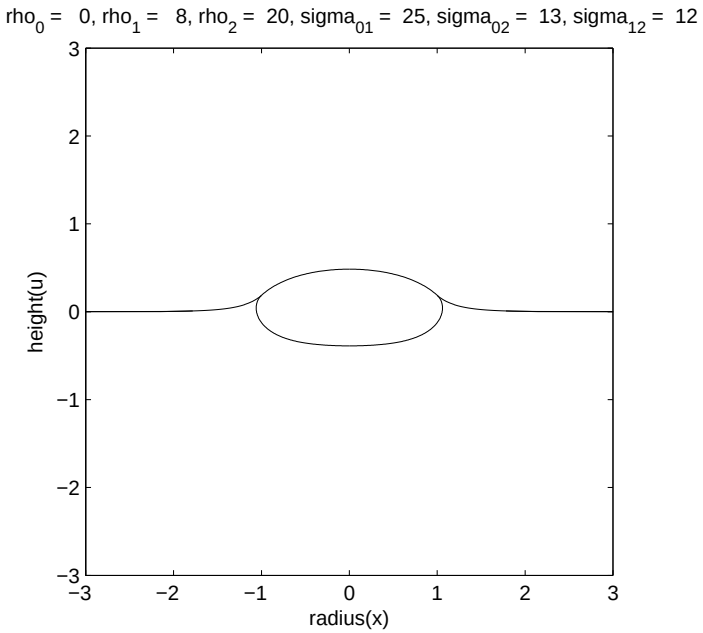

Figure 4: Drop with Tangential Surfaces

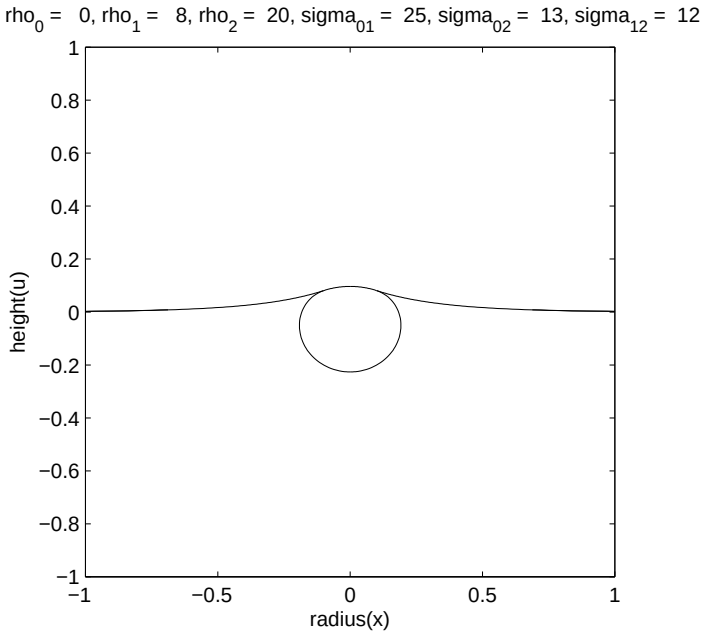

Figure 5: Small Drop

5 Acknowledgements

I would like to thank Ray Treinen for his extensive advice and guidance on this research project and provision of the previous research referenced. I would also like to thank Robert Finn for his gracious input on the results.

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