HOLONOMIZING AND DEHOLONOMIZING: The Pivotal Role of Commutators in Action

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Abstract

We develop a Lyapunov-based method to construct control equations for a non-tilting disk that is rolling without slipping. Given two control parameters, and five state-space variables, we use Lie brackets to effectively "holonomize" a non-holonomic system. Then we convert this system of artificially actuated controls into a time-dependent system of real controls. This is a simplified version of the unicycle control problem.

1 Introduction

Introduction. The majority of the developments in control theory to date have focused on the control of holonomic systems. Nonholonomic systems are fundamentally different and present unique problems. Our goal is to develop a control law for a real unicycle – not the non-tipping, 3-degrees-of-freedom wheeled robot often called a unicycle in the literature. In order to develop a mathematical foundation on which to build our control for this complicated problem, we first investigated the behavior of a simpler system that we call "masking tape". The masking tape may roll about the x-y

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pose.jpg 

Figure 1: The Unicycle: simple, fun, and most difficult to stabilize.

plane without slipping, and may pivot about its point of contact. However, the masking tape is prevented from tipping. We develop a time-dependent feedback control for the masking tape, keeping in mind our greater goal of stabilizing the unicycle.

2 Masking Tape

Our non-tilting rolling wheel, henceforth referred to as masking tape, moves by two active torques, Γ_ω and Γ_μ, which directly control the angular acceleration of rolling and turning, respectively. The state-space variables are defined by the vector

$$\mathbf{x} := \begin{pmatrix} x \\ y \\ \psi \\ \omega \\ \mu \end{pmatrix} \tag{1}$$

where (x, y) is the point of contact of the wheel with the horizontal plane. The angle ψ measures the direction of forward motion with respect to the x-axis, and ω, μ are the respective angular velocities of rolling and turning, as shown in Figure (2).

The constraint of rolling without slipping is given by the two equations

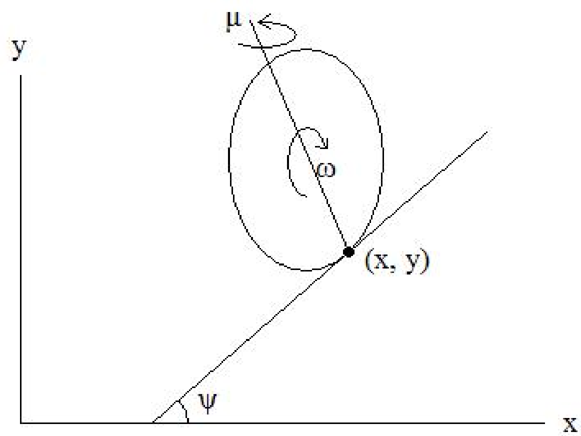

Figure 2: Masking tape system, ref[1]

$$\begin{aligned} \dot{x} &= R\omega cos\psi \\ \dot{y} &= R\omega sin\psi \end{aligned} \tag{2}$$

where R, the radius of the tape, we will conveniently define to equal 1. The state-space "velocity" vector can be expressed as the following function of state-space variables, $\mathbf{x}$, and controls, $\mathbf{u} = (\Gamma_\omega, \Gamma_\mu)$:

$$\mathbf{f}(\mathbf{x}, \mathbf{u}) := \dot{\mathbf{x}} := \begin{pmatrix} \dot{x} \\ \dot{y} \\ \dot{\psi} \\ \dot{\omega} \\ \dot{\mu} \end{pmatrix} = \begin{pmatrix} \omega cos\psi \\ \omega sin\psi \\ \mu \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} \Gamma_\omega + \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \Gamma_\mu \tag{3}$$

A key point is that the constraints make this system *nonholonomic*. This means that there is a restriction on the instantaneous direction of motion, even though there exists a path to every point in state-space. The tape cannot slide 1 unit sideways, but it can follow a curved path that will eventually bring it to that position. Physical systems associated with nonholonomity are ones that involve any type of rotation. A sailboat, for example, can move in a zigzag pattern against the wind. In engineering, nonholonomity is often related to systems that are underactuated (have more joints than controls). We will have much more to say about nonholonomity later on.

3 Minimizing the Value Function

The fundamental goal of control theory is to find an optimal set of controls for a system. A natural first step is to define what we mean by *o*ptimal. In our case, we wish to bring the masking tape to some point in state-space (which we define as the origin of our coordinate system), on a short path but without using too much power. Let us begin by defining a *cost function* that tallies up the quantities which we want to minimize over some interval of time:

$$J(\mathbf{x}(t_f)) := \int_{t_i}^{t_f} L(\mathbf{x}, \dot{\mathbf{x}}, t)dt \tag{4}$$

For the tape, we have defined the typical LQR (Linear Quadratic Regulator [1]) cost integrand

$$L(\mathbf{x}, \dot{\mathbf{x}}, t) := \frac{1}{2}(\mathbf{x}^T Q\mathbf{x} + \mathbf{u}^T R\mathbf{u}) \tag{5}$$

and for the purpose of this paper we choose the identity matrices $Q := I_5$, $R := I_2$. It costs us to have the tape spend time far from the origin, and it also costs us to use our controls, so an optimal solution must compromise between path length and energy spent.

Definition The *value function* is the minimal cost function with respect to the control parameters $\mathbf{u}$, on the time interval $[t, t_f]$.

$$V(\mathbf{x}, t) := \min_{\mathbf{u}} \left[J_{t,t_f}(\mathbf{x}, \mathbf{u}(\mathbf{x}, t)) \right] \tag{6}$$

Note that we are minimizing with respect to $\mathbf{u}$ because this is the parameter that we can change, but the quantity being minimized contains both $\mathbf{u}$ and $\mathbf{x}$ - which changes as a function of $\mathbf{u}$. The value function corresponds to the cost of the optimal solution, and discovering the value function of a system is one method of finding an optimal control.

The classic method for finding the value function is to solve the Hamilton-Jacobi-Bellman partial differential equation [1]. Unfortunately, in a nonlinear system (such as masking tape), this PDE might not be analytically solvable. This is a major problem because implementing a numerical method requires an initial value for the value function, which is not known.
An alternate approach involves guessing at some value function which might be different than the optimal one, but is guaranteed to asymptotically stabilize the system, i.e. make our masking tape always approach the origin as time increases.

4 Control Lyapunov Functions

Definition Suppose a function $W(\mathbf{x})$ fulfills the following conditions:

$$\begin{gathered}\mathbf{x} = 0 \Rightarrow W = 0 \\ \mathbf{x} \neq \mathbf{0} \Rightarrow W > 0 \\ \lim_{|\mathbf{x}| \to \infty} W = \infty\end{gathered} \tag{7}$$

Then for W to asymptotically stabilize the system to the origin, we require that, unless $\mathbf{x} = \mathbf{0}$, there exists a choice of control $\mathbf{u}$ such that $W(\mathbf{x})$ is decreasing. Equivalently,

$$\inf_{\mathbf{u}} \left[\nabla W \cdot \mathbf{f}(\mathbf{x}, \mathbf{u})\right] < 0 \qquad \forall \mathbf{x} \neq \mathbf{0} \tag{8}$$

A function W which fulfills the conditions above is called a *control Lyapunov function* (clf).

The dot-product between two vectors is negative when the angle between them is greater than $\frac{\pi}{2}$. Since, by definition, the gradient ∇W always points upwards (or in the direction of greatest increase), condition (8) says that $\mathbf{f}(\mathbf{x}, \mathbf{u}) := \dot{\mathbf{x}}$ is always pointing at some angle below horizontal. In other words, the components of the vector $\dot{\mathbf{x}}$ are always negative, so that $\mathbf{x}$ (and therefore $W(\mathbf{x})$) is decreasing in time.
We can think of this graphically as a multi-dimensional paraboloid-type surface upon which our velocity vector always points along some downhill trajectory (see Figure 3 below). Note that if the dot product were identically

zero, i.e. the angle between the vectors equals $\frac{\pi}{2}$, the $\dot{\mathbf{x}}$ would be horizontal and this would correspond to a level curve on the graph $W(\mathbf{x})$.

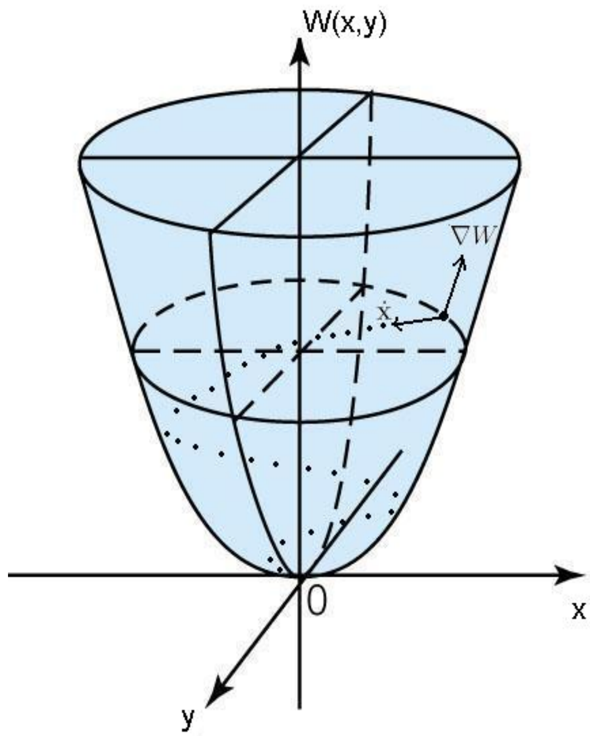

Figure 3: clf graphical interpretation - the dotted line represents a trajectory asymptotically stabilizing to the origin. ref[1]

A typical form of clf is $W(\mathbf{x}) = \frac{1}{2}\mathbf{x}\cdot\mathbf{x}$. This conveniently makes $\nabla W = \mathbf{x}$. We chose this clf for the masking tape, despite the knowledge that there are some points in our state-space where this clf does not work. For example, when the masking tape is rolling directly away from the origin, it does not know which way to turn. The feedback control law must have a discontinuity, or else not be a pure feedback law, but have a time dependence as well. We will see that this is a general problem associate with nonholonomic systems. Let us put this point aside for now and continue the clf method.

4.1 Finding a Feedback Law

E. Sontag [3] suggests the following method of choosing a feedback control law for a given a clf:
Define the generalized velocity function $\mathbf{f}(\mathbf{x}, \mathbf{u}) = \dot{\mathbf{x}}$ in the general form

$$\mathbf{f}(\mathbf{x}, \mathbf{u}) := \mathbf{h}(\mathbf{x}) + u_1\mathbf{g}_1(\mathbf{x}) + u_2\mathbf{g}_2(\mathbf{x}) + ... + u_n\mathbf{g}_n(\mathbf{x}) \tag{9}$$

where the u_i are the control parameters. For convenience, let us rename the terms of $\nabla W \cdot \mathbf{f}(\mathbf{x}, \mathbf{u})$ (from equation (8)) as

$$\begin{cases} a := a(x) := \nabla W \cdot \mathbf{h}(\mathbf{x}) \\ b_i := b_i(x) := \nabla W \cdot \mathbf{g}_i(\mathbf{x}) \\ \beta := \sum_{i=1}^{n} {b_i}^2 \end{cases} \tag{10}$$

The clf condition in equation (8) can be expressed as the relationship

$$\beta = 0 \quad \Rightarrow \quad a < 0 \tag{11}$$

for every nonzero $\mathbf{x}$. This condition is equivalent to the statement that when the system is not controllable, (unless we are at the origin) we are still headed towards the origin . If we choose a feedback law of the form

$$u_i = -b_i \frac{a + \sqrt{a^2 + \beta^2}}{\beta} \tag{12}$$

then for $\beta \neq 0$, equation (8) reduces to

$$-\sqrt{a^2 + \beta^2} < 0 \tag{13}$$

which is true, so the clf requirement is fulfilled. When $\beta = 0$ we want to define $u_i = 0$ for all i. If condition (11) is fulfilled, then the above choice of control law is continuous in the limit where $\beta \to 0$.

Applying this formula to the masking tape system, using $\nabla W = \mathbf{x}$, we make the following definitions:

$$
\begin{aligned}
a &:= x\omega\cos\psi + y\omega\sin\psi + \psi\mu \\
b_1 &:= \omega \\
b_2 &:= \mu \\
\beta &:= \omega^2 + \mu^2
\end{aligned} \tag{14}
$$

Equation (12) gives us the feedback control law

$$
\begin{aligned}
\Gamma_\omega &= \begin{cases} -\omega\frac{(x\omega\cos\psi+y\omega\sin\psi+\psi\mu)+\sqrt{(x\omega\cos\psi+y\omega\sin\psi+\psi\mu)^2+(\omega^2+\mu^2)^2}}{\omega^2+\mu^2} & \omega^2+\mu^2\neq 0 \\ 0 & \omega^2+\mu^2=0 \end{cases} \\
\Gamma_\mu &= \begin{cases} -\mu\frac{(x\omega\cos\psi+y\omega\sin\psi+\psi\mu)+\sqrt{(x\omega\cos\psi+y\omega\sin\psi+\psi\mu)^2+(\omega^2+\mu^2)^2}}{\omega^2+\mu^2} & \omega^2+\mu^2\neq 0 \\ 0 & \omega^2+\mu^2=0 \end{cases}
\end{aligned} \tag{15}
$$

If this feedback law fulfills the condition (11), then the controls are continuous. Unfortunately, it does not pass the test. When $\beta = 0$ we have $a =$. This is not surprising, because we concluded earlier that it is not possible to have a continuous stabilizing feedback law for masking tape.
Suppose that the tape starts at rest at some point away from the origin. Since the angular velocities ω and μ both equal zero, the feedback control law above says do absolutely nothing! The reason for this troubling phenomenon lies in the nonholonomic property of the system.

5 More on Non-Holonomity

A consequence of non-holonomity is that operations are not commutative: 'Turn and then roll' is *n*ot equivalent to 'roll and then turn'. This non-commutative property leads to the mathematical definition of non-holonomity. Let us first rewrite each column of equation (3) as differential operator,

$$
D = \mathbf{D}_1 + \Gamma_\omega \mathbf{D}_2 + \Gamma_\mu \mathbf{D}_3 \tag{16}
$$

where each $\mathbf{D}_i$ is composed of partial differential operators:

$$\begin{aligned}
\mathbf{D}_1 =& \omega \cos\psi \partial_{\mathbf{x}} + \omega \sin\psi \partial_{\mathbf{y}} + \mu \partial_{\psi} \\
\mathbf{D}_2 =& \partial_{\omega} \\
\mathbf{D}_3 =& \partial_{\mu}
\end{aligned} \tag{17}$$

We can think of ∂_x as the unit column vector with a 1 in the x place and zeros everywhere else. An analogous description goes for the rest of the partial differential operators. With this description in mind, we see the differential operator as a vector field in state-space.
Let us utilize the operator notation to express two closely related definitions from *Lie Algebra*:

Definition (I) Given two differential operators $\mathbf{D}_1, \mathbf{D}_2$, define the *Lie-bracket*

$$[\mathbf{D}_1, \mathbf{D}_2] := \mathbf{D}_1\mathbf{D}_2 - \mathbf{D}_2\mathbf{D}_1$$

.

Definition (II) The *commutator* of two differential operators

$$(\mathbf{D}_1, \mathbf{D}_2) := \mathbf{D}_2^{-1}\mathbf{D}_1^{-1}\mathbf{D}_2\mathbf{D}_1$$

Lemma 5.1. *The commutator of two differential operators equals the identity (meaning the two operations are commutative), if and only if the Lie-bracket equals zero.*

Proof. Suppose $\mathbf{D}_1\mathbf{D}_2 - \mathbf{D}_2\mathbf{D}_1 = 0$.
Then

$$\begin{aligned}
\mathbf{D}_2\mathbf{D}_1 =& \mathbf{D}_1\mathbf{D}_2 \\
{\mathbf{D}_1}^{-1}\mathbf{D}_2\mathbf{D}_1 =& \mathbf{D}_2 \\
{\mathbf{D}_2}^{-1}{\mathbf{D}_1}^{-1}\mathbf{D}_2\mathbf{D}_1 =& I
\end{aligned}$$

We see that the converse holds by the exact same steps in the reveres order.
Therefore

$$\mathbf{D}_1\mathbf{D}_2 - \mathbf{D}_2\mathbf{D}_1 = 0 \quad \Longleftrightarrow \quad {\mathbf{D}_2}^{-1}{\mathbf{D}_1}^{-1}\mathbf{D}_2\mathbf{D}_1 = I$$

□

If two operators do not commute, then the Lie-bracket returns another differential operator, that corresponds to a motion created by the commutation of the two.

Definition A system is said to be *holonomic* if its differential operators span the space generated by the Lie-bracket.

If the Lie-bracket produces new differential operators that are independent of the original ones, then the system is nonholonomic.
Now we can understand a mathematical formulation of why this masking tape is being difficult. It belongs to the class described in Brockett's theorem.

Theorem 5.2. (Brockett's Theorem)*A nonholonomic control system cannot be asymptotically stabilized to a resting configuration by smooth time-invariant control laws.[2]*

6 Holonomizing the System

Lie Brackets for our system produce three independent differential operators in addition to the original three:

$$\begin{aligned} \mathbf{D_4} :=&[\mathbf{D}_1, \mathbf{D}_2] = \cos\psi\partial_{\mathbf{x}} + \sin\psi\partial_{\mathbf{y}} \\ \mathbf{D_5} :=&[\mathbf{D}_1, \mathbf{D}_3] = \partial_{\psi} \\ \mathbf{D_6} :=&[\mathbf{D_4}, \mathbf{D_5}] = \sin\psi\partial_{\mathbf{x}} - \cos\psi\partial_{\mathbf{y}} \end{aligned} \tag{18}$$

By adding these three operators to our original system, we can effectively "holonomize" the system. The generalized velocity function for our holonomized system becomes

$$\mathbf{f} = \begin{pmatrix} \omega cos\psi \\ \omega sin\psi \\ \mu \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} \Gamma_\omega + \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \Gamma_\mu + \begin{pmatrix} \cos\psi \\ \sin\psi \\ 0 \\ 0 \\ 0 \end{pmatrix} u_1 + \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} u_2 + \begin{pmatrix} \sin\psi \\ -\cos\psi \\ 0 \\ 0 \\ 0 \end{pmatrix} u_3 \tag{19}$$

where now $\mathbf{f} = \mathbf{f}(\mathbf{x}, \mathbf{u}, u_1, u_2, u_3)$.
We can pretend to actually have the controls u_1, u_2, u_3, and develop a control law with Sontag's formula. Define the clf-condition variables

$$
\begin{aligned}
a &:= x\omega\cos\psi + y\omega\sin\psi + \psi\mu \\
b_1 &:= \omega \\
b_2 &:= \mu \\
b_3 &:= x\cos\psi + y\sin\psi \\
b_4 &:= \psi \\
b_5 &:= x\sin\psi - y\cos\psi \\
\beta &:= \omega^2 + \mu^2 + x\cos\psi + y\sin\psi^2 + \psi^2 + x\sin\psi - y\cos\psi^2
\end{aligned}
$$

Let us also define the term that is common to all the control functions,

$$\Phi(a(\mathbf{x}), \beta(\mathbf{x})) := \frac{a + \sqrt{a^2 + \beta^2}}{\beta} \tag{20}$$

The holonomized feedback law for $\beta \neq 0$, which we call $\tilde{\mathbf{u}}$, can be conveniently written as follows:

$$\tilde{\mathbf{u}} := \begin{pmatrix} \Gamma_\omega \\ \Gamma_\mu \\ u_1 \\ u_2 \\ u_3 \end{pmatrix} = - \begin{pmatrix} \omega \\ \mu \\ x\cos\psi + y\sin\psi \\ \psi \\ x\sin\psi - y\cos\psi \end{pmatrix} \Phi(a(\mathbf{x}), \beta(\mathbf{x})) \tag{21}$$

For $\beta = 0$ we define $\tilde{\mathbf{u}} := 0$. Notice that in this case $\beta = 0$ only at the origin, $\mathbf{x} = \mathbf{0}$. This means that the origin is the only place where $\tilde{\mathbf{u}} = 0$. The system fulfills the clf condition (11) by default because for any point $\mathbf{x}$ outside of the origin, β is never zero.

We have holonomized our masking tape system. Simulations show that the feedback law stabilizes the system quite optimally. However, this is a quite artificial solution. In order to make the system holonomic, fully actuated, and thus stabilizable, we tossed away the rolling constraints and invented imaginary controls that shift the tape in any direction on the plane. This hardly seems like an effective solution.

Nevertheless, recall that the fake controls were developed by taking the span of the real controls. Note the similarity between the form 'turn left, roll forward, turn right, roll back', and the commutator $\mathbf{D}_2^{-1}\mathbf{D}_1^{-1}\mathbf{D}_2\mathbf{D}_1$. We use the commutator to express each of our fake controls, u_1, u_2, u_3, as a series of operations done the real controls, Γ_ω and Γ_μ. This is how we effectively *deholonomize* the system.

7 Deholonomizing

The holonomized system exhibits "nice" behavior, so an attempt to implement the fake controls will be made. Remembering that u_2 is the commutation of the constraints with Γ_μ and abusing the Lie Bracket, we consider the constraints to be an identity and say that $u_2 = [1, D_2]$. It is then desired to find a way to perform $D_2 D_2^{-1}$. In other words, we must search for a function $\gamma_2(\lambda_2, t, T)$ such that

$$\Gamma_\mu = \gamma_2(\lambda_2, t, T) \tag{22}$$

has the same effect on ψ as

$$u_2(t) = \lambda_2 \tag{23}$$

at time t, whenever t is a multiple of some period T. In effect, this force will act like u_2 applied at a constant magnitude. It is clear that the step function

$$\gamma_2(\lambda_2, t, T) = \begin{cases} \frac{4\lambda_2}{T} & 0 < t < \frac{T}{2} \\ -\frac{4\lambda_2}{T} & \frac{T}{2} < t < T \end{cases}$$

satisfies these requirements. Another satisfactory $\gamma_2(\lambda_2, t, T)$ is given by

$$\gamma_2(\lambda_2, t, T) = \frac{2\pi\lambda_2}{T} \sin\left(\frac{2\pi t}{T}\right) \tag{24}$$

The period T plays a crucial role in the effectiveness of this technique. The smaller T is, the better our approximation that $u_2(t)$ is constant, and the more our impementation of u_2 looks like it is supposed to. However, we notice that as T approaches zero, the magnitude of the torques that must be applied become infinite. We must therefore settle on some intermediate value for T that will be implementable in a real system.

The same type of argument holds for the implementation of u_1. However, u_3 is not a first order Lie-bracket, but is given by

$$[[\mathbf{D}_1, \mathbf{D}_2], [\mathbf{D}_1, \mathbf{D}_3]] \tag{25}$$

Thus we seek functions $\phi_1(\lambda_3, t, T)$ and $\phi_2(\lambda_3, t, T)$ such that

$$u_1 = \phi_1(\lambda_3, t, T) \tag{26}$$

and

$$u_2 = \phi_2(\lambda_3, t, T) \tag{27}$$

have the same effect on x and y as

$$u_3(t) = \lambda_3 \tag{28}$$

at time t, whenever t is a multiple of some period T. When searching for these functions, we treat u_1 and u_2 as real controls that can be implemeted continously. One example of satisfactory functions is:

$$\phi_1(\lambda_3, t, T) = \begin{cases} 0 & 0 < t < \frac{T}{8} \\ 4\lambda_3 & \frac{T}{8} < t < \frac{3T}{8} \\ 0 & \frac{3T}{8} < t < \frac{5T}{8} \\ -4\lambda_3 & \frac{5T}{8} < t < \frac{7T}{8} \\ 0 & \frac{7T}{8} < t < T \end{cases} \qquad \phi_2(\lambda_3, t, T) = \begin{cases} \frac{4\pi}{3T} & 0 < t < \frac{T}{8} \\ 0 & \frac{T}{8} < t < \frac{3T}{8} \\ -\frac{4\pi}{3T} & \frac{3T}{8} < t < \frac{5T}{8} \\ 0 & \frac{5T}{8} < t < \frac{7T}{8} \\ \frac{4\pi}{3T} & \frac{7T}{8} < t < T \end{cases} \tag{29}$$

Another example is

$$\phi_1(\lambda_3, t, T) = \zeta_1 \cos(\frac{2\pi t}{T}), \qquad \phi_2(\lambda_3, t, T) = \zeta_2 \sin(\frac{2\pi t}{T}) \tag{30}$$

where ζ_1 and ζ_2 are functions of λ_3 and T.

8 conclusion

We have taken a nonholonomic system and artificially "completed" it, by implementing a set of fake controls that span the entire space generated by the Lie-bracket. This allowed us to make use of a control Lyapunov function for our feedback law, which was now smooth but with little resemblance to the real system. Making use of commutators, we deholonomized and set up a smooth control law based on a temporal oscillation in force. We are at present time still testing variations of this time-varying technique, to find out just how optimal a method this is. The next step is trying this process on a unicycle, and see if it actually runs.

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