COMPARISON OF INVERSION TECHNIQUES FOR ILL-CONDITIONED MATRICES

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Abstract: *This paper summarizes the work under the direction of Prof. A.G.Ramm at the 2006 session of the KSU SUROP / REU "Brainstorming and Barnstorming". A background of ill-posed problems and current methods - specifically variational regularization - used to numerically solve them are given before introducing two new methods and comparing them against variational regularization. The experiments are performed using the $N \times N$ Hilbert Matrix H_N ($H_{i,j} = (1+i+j)^{-1}, i, j \in \{\mathbb{R} : [1, N]\}$) due to its property of being ill-conditioned for $N > 6$.*

1. Introduction

Ill-posed problems were first introduced in 1902 by Hadamard [1].

Definition 1.1. [1] A *well-posed problem* $Au = f$, where $A : X \to X, X$ is a Banach space, is a problem which satisfies the following:

- A solution exists in X
- The solution is unique in X
- The solution depends continuously in X on the data f.

Definition 1.2. [1] An *ill-posed problem* is a problem which is not well-posed.

Many practical problems are reduced to solving LAS (linear algebraic systems) which are ill-conditioned, so the creation of a most efficient method for solving ill-posed problems are of great practical importance.

2. Ill-Posed Problems

For the purposes of this paper, we will think of an ill-posed problem as a LAS which contains an ill-conditioned matrix.

Definition 2.1. The *condition number* κ of a matrix A is $||A|| \cdot ||A^{-1}||$.

Definition 2.2. [4] An *ill-conditioned matrix* is some matrix with a large condition number.

The difficulty with ill-posed problems is that given an LAS $Au = f_\delta$ such that $f_\delta = f + \eta$ where η is some noise, which satisfies $||f_\delta - f|| < \delta$

and the condition number of A is large, small perturbations of δ may result in a very large change in u.

Example 2.3. Suppose $Au = f$ is the LAS needed to be solved, where:

$$A := H_3 = H_{i,j} = (1 + i + j)^{-1} \qquad u_i := i; \qquad \delta = 0.01$$

Since $Au = f$ then $f^T \approx (\ 3.00000 \quad 1.91667 \quad 1.43333\ )$ and

$$f_\delta = f + \delta \frac{e}{||e||} \approx (\ 3.00198 \quad 1.92280 \quad 1.43241\ )^T,$$

where e is some normalized 3×1 matrix whose entries have normal distribution, mean 0 and standard deviation 1.

So now,

$$Au_\delta = f_\delta \implies u_\delta = A^{-1} f_\delta \approx (\ 0.7695 \quad 3.2714 \quad 1.7903\ )^T.$$

The relative error of this u_δ, $\frac{||u_\delta - u||}{||u||}$ is 139.49%

The condition number of A in the example is approximately 524, which is why the error is not very large.

Claim 2.4. *As the condition number of some matrix A increases, the upper bound of the error increases.*

Proof. If $Au = f$ represents some Linear Algebraic System where A is the $n \times n$ matrix, u is the variable matrix and f is the solution matrix, and $\kappa = ||A|| \cdot ||A^{-1}||$ is the condition number of A,

$$||u_\delta - u|| = ||A^{-1} f_\delta - A^{-1} f|| = ||A^{-1}(f_\delta - f)|| \leq ||A^{-1}||\delta = \frac{\kappa}{||A||}\delta. \tag{2.1}$$

□

From 2.1, we can see that as the condition number of A becomes large, the residual $||u_\delta - u||$ also becomes large. Because of this, we can use the condition number of A to estimate the expected magnitude of the error.

3. Newer Methods of Inversion

We will discuss and compare two newer methods of inversion including the dynamical systems method (DSM) from [4] and a method of minimizing the upper bound of error (UBM).

3.1. **Upper Bound Minimization.**

The first method we will discuss is Upper Bound Minimization. This method simply finds an upper bound of $||T_a^{-1}A^*f_\delta - u||$ and minimizes it with respect to a. In this way, it is possible to have a very small calculation time: the method requires only one equation to be solved before plugging the result into a formula, though since the upper bound is not an equality, the relative error will in general be greater than with other methods, though this will be shown in detail in the results section.

Given some $N \times N$ matrix A and $N \times 1$ matrix f such that:

$$Au = f \qquad \text{Let } T := A^*A, \qquad ||f - f_\delta|| \leq \delta,$$

$$\text{and } u_{a,\delta} := (T + aI)^{-1}A^*f_\delta := T_a^{-1}A^*f_\delta,$$

where I is the identity matrix of size $N \times N$. Then

$$||u_{a,\delta} - u|| = ||T_a^{-1}A^*f_\delta - u|| \leq ||T_a^{-1}A^*(f_\delta - f)|| + ||T_a^{-1}A^*f - u||.$$

From [4], $||T_a^{-1}A^*|| \leq (2\sqrt{a})^{-1}$ if $a > 0$, so

$$||T_a^{-1}A^*(f_\delta - f)|| + ||T_a^{-1}A^*f - u|| \leq \frac{\delta}{2\sqrt{a}} + a||T_a^{-1}u||.$$

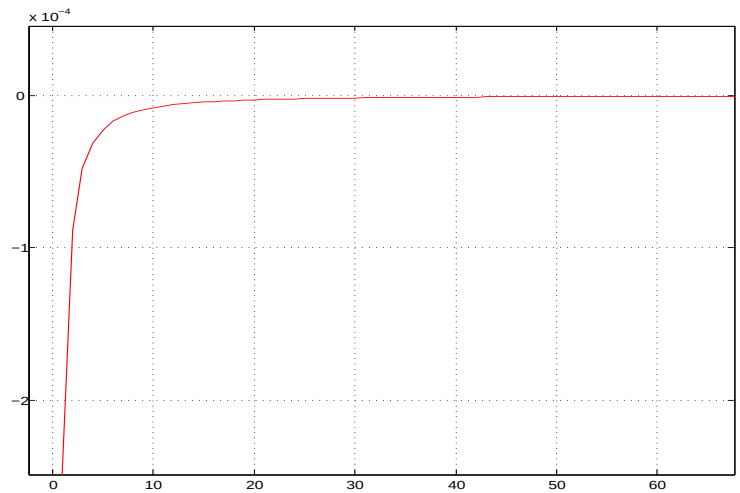
Further, $||T_a^{-1}|| \leq \frac{1}{a+\lambda_n}$, so

$$\varepsilon := \frac{\delta}{2\sqrt{a}} + \frac{a}{a + \lambda_n^2}||u||.$$

We want to set $\varepsilon(a) := \varepsilon$ and minimize ε with respect to a:

$$\frac{\partial \varepsilon}{\partial a} := \dot{\varepsilon} := -\frac{\delta}{4a^{3/2}} + \frac{\lambda_n^2}{(a + \lambda_n^2)^2}||u|| = 0. \tag{3.1}$$

Graph for solution of a vs. $\dot{\varepsilon}$:



So the Upper Bound Minimization method will consist of:

$$u_{a,\delta} = (T_a^{-1} A^*) f_\delta$$

where a is selected by solving equation (3.1). A major limitation of this method is that it requires prior knowledge of $||u||$.

3.2. **Dynamical Systems Method.**

The Dynamical Systems Method is defined in [4] as the solution to the following initial value problem:

$$\dot{u} = -u + T_{a(t)}^{-1} A^* f \qquad u(0) = u_0 \tag{3.2}$$

where $f = Ay$, $y \perp N$, and $a(t) > 0$ is a monotonically decaying function such that $\lim_{t\to\infty} a(t) = 0$ and $\int_0^\infty a(t)\ dt = \infty$, and u_0 is an arbitrary initial approximation.

The method we have used to solve this IVP is Runge-Kutta (RK) of order 5 (RK5), created by Fehlberg. This version of RK includes an embedded RK of order 4 (RK4) which allows us to compare the two orders for error at each step. As a result, the step size for each iteration can be dynamically increased or decreased depending on the ratio of the two orders. With our software, we alter the step size h by:

$$y_{error} = ||y_5 - y_4|| \qquad h_{n+1} = h_n \tfrac{e_{allowed}}{y_{error}}.$$

Where y_4 and y_5 are the results of RK4 and RK5, respectively and $e_{allowed}$ is the maximum allowed error. If greater accuracy is needed,

decreasing $e_{allowed}$ will accomplish this, though only slightly and at great cost of computation time.

Rather than using the original Fehlberg parameters, we will use the Cash-Karp parameters due to the better efficiency in these parameters [3].

The stopping time for this RK5 method will use the discrepancy principle: the program will run until

$$||Au_\delta - f_\delta|| < c\delta \tag{3.3}$$

such that $1 < c < 2$.

DSM3 with RK5 can have a great accuracy, though the running time will increase compared to simpler problems due to the complexity of solving a differential equation IVP. These results will be discussed further in the results section.

In our experiments, we will take $c = 1.8$ in 3.2 and choose $a(t)$ in 3.3 as follows:

$$c = 1.8 \quad a(t) = \exp[(t + .001)^{-4.5}] - 1$$

where h is the current step size.

3.3. **Variational Regularization.**

Variational regularization (VR) is one of the currently used methods of solving ill-conditioned linear algebraic systems. We will use this method to compare against the newer methods presented in this paper. For the convenience of the reader, VR is briefly described here [2], [4].

In this method, the solution of the ill-conditioned linear system is obtained by solving the following optimization problem:

$$\min_u \left\{ ||Au - f_\delta||^2 + a||u||^2 \right\},$$

where a is a regularization parameter.

In this paper, we will use the Newton method to find a from the discrepancy equation:

$$||Au_{a,\delta} - f_\delta|| = c\delta \qquad c = 1.8.$$

This method is efficient and yields the approximate solution to the equation $Au = f$ which we use for comparison with the solutions obtained by the methods (3.1) and (3.2).

4. Experiments

For all of the following experiments, the solution of the equation $Au = f$ was approximated, given the data f_δ, δ, A.

$$A = H_n = \begin{pmatrix} 1 & \frac{1}{2} & \cdots & \frac{1}{n} \\ \frac{1}{2} & \frac{1}{3} & \cdots & \frac{1}{n+1} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{1}{n} & \frac{1}{n+1} & \cdots & \frac{1}{2n-1} \end{pmatrix},$$

and the stopping time in the DSM method was calculated using the discrepancy principle:

$$||Au - f_\delta|| < c\delta, \qquad 1 < c < 2,$$

where $c = 1.8$ and $u^T = [u_1 \dots u_n]$, with u_i and δ provided in the result tables. The results were calculated using software written using The Mathworks, Inc. application MATLAB®.

The following is the order of results:
· Results of UBM
· Results of DSM3
· Results of VR
· Comparison of UBM and VR by error
· Comparison of DSM3 and VR by error
· Comparison of UBM and VR by computation time
· Comparison of DSM3 and VR by computation time

4.1. **Results of UBM.**

u_i	δ	cputime (s)	Error $\frac{\|\|u_{est}-u_{exact}\|\|}{\|\|u_{exact}\|\|}$	Error $\frac{\|\|A^{-1}f_\delta-u\|\|}{\|\|u\|\|}$
1	0.001	0.078	5.320e-001	8.964e+013
1	0.01	0.016	6.647e-001	8.964e+014
1	0.1	0.031	7.645e-001	8.964e+015
i	0.001	0.031	8.002e-001	1.541e+012
i	0.01	0.031	8.719e-001	1.541e+013
i	0.1	0.031	9.161e-001	1.541e+014
$\sin i$	0.001	0.047	9.933e-001	1.264e+014
$\sin i$	0.01	0.031	9.954e-001	1.264e+015
$\sin i$	0.1	0.031	9.968e-001	1.264e+016

TABLE 1. $A = H_{100}$, $\text{cond}(A) \approx 3.78 \times 10^{150}$

u_i	δ	cputime (s)	Error $\frac{\|\|u_{est}-u_{exact}\|\|}{\|\|u_{exact}\|\|}$	Error $\frac{\|\|A^{-1}f_\delta-u\|\|}{\|\|u\|\|}$
1	0.001	0.000	5.353e-001	5.572e+013
1	0.01	0.016	6.634e-001	5.572e+014
1	0.1	0.031	7.625e-001	5.572e+015
i	0.001	0.000	7.994e-001	1.902e+012
i	0.01	0.031	8.716e-001	1.902e+013
i	0.1	0.031	9.151e-001	1.902e+014
$\sin i$	0.001	0.031	9.867e-001	7.862e+013
$\sin i$	0.01	0.031	9.909e-001	7.862e+014
$\sin i$	0.1	0.031	9.938e-001	7.862e+015

TABLE 2. $A = H_{50}$, $\text{cond}(A) \approx 1.42 \times 10^{74}$

Results of UBM, cont.

u_i	δ	cputime (s)	Error $\frac{\|\|u_{est}-u_{exact}\|\|}{\|\|u_{exact}\|\|}$	Error $\frac{\|\|A^{-1}f_\delta-u\|\|}{\|\|u\|\|}$
1	0.001	0.031	5.376e-001	4.472e+013
1	0.01	0.016	6.569e-001	4.472e+014
1	0.1	0.031	7.598e-001	4.472e+015
i	0.001	0.031	8.006e-001	3.733e+012
i	0.01	0.031	8.682e-001	3.733e+013
i	0.1	0.000	9.108e-001	3.733e+014
$\sin i$	0.001	0.000	9.688e-001	6.232e+013
$\sin i$	0.01	0.031	9.770e-001	6.232e+014
$\sin i$	0.1	0.031	9.835e-001	6.232e+015

TABLE 3. $A = H_{20}$, $\text{cond}(A) \approx 2.45 \times 10^{28}$

u_i	δ	cputime (s)	Error $\frac{\|\|u_{est}-u_{exact}\|\|}{\|\|u_{exact}\|\|}$	Error $\frac{\|\|A^{-1}f_\delta-u\|\|}{\|\|u\|\|}$
1	0.001	0.031	5.355e-001	6.991e+012
1	0.01	0.000	6.532e-001	6.991e+013
1	0.1	0.016	7.568e-001	6.991e+014
i	0.001	0.031	8.005e-001	7.689e+011
i	0.01	0.031	8.652e-001	7.689e+012
i	0.1	0.031	9.080e-001	7.689e+013
$\sin i$	0.001	0.000	9.552e-001	9.652e+012
$\sin i$	0.01	0.000	9.644e-001	9.652e+013
$\sin i$	0.1	0.016	9.733e-001	9.652e+014

TABLE 4. $A = H_{15}$, $\text{cond}(A) \approx 6.12 \times 10^{20}$

4.2. **Results of DSM.**

u_i	δ	cputime (s)	iterations	Error $\frac{\|\|u_{est}-u_{exact}\|\|}{\|\|u_{exact}\|\|}$	Error $\frac{\|\|A^{-1}f_\delta-u\|\|}{\|\|u\|\|}$
1	0.001	0.250	13	3.011e-002	8.884e+013
1	0.01	0.219	12	1.702e-001	8.884e+014
1	0.1	0.313	11	1.112e+000	8.884e+015
i	0.001	1.750	55	3.448e-002	1.527e+012
i	0.01	1.547	39	6.660e-002	1.527e+013
i	0.1	1.359	31	1.089e-001	1.527e+014
$\sin i$	0.001	0.250	11	9.766e-001	1.253e+014
$\sin i$	0.01	0.219	10	9.986e-001	1.253e+015
$\sin i$	0.1	0.031	2	9.906e-001	1.253e+016

TABLE 5. $A = H_{100}$, $\text{cond}(A) \approx 3.78 \times 10^{150}$

u_i	δ	cputime (s)	iterations	Error $\frac{\|\|u_{est}-u_{exact}\|\|}{\|\|u_{exact}\|\|}$	Error $\frac{\|\|A^{-1}f_\delta-u\|\|}{\|\|u\|\|}$
1	0.001	0.125	13	3.439e-002	5.570e+013
1	0.01	0.063	12	2.200e-001	5.570e+014
1	0.1	0.063	11	1.526e+000	5.570e+015
i	0.001	0.109	29	4.558e-002	1.901e+012
i	0.01	0.078	19	8.658e-002	1.901e+013
i	0.1	0.063	17	1.124e-001	1.901e+014
$\sin i$	0.001	0.047	11	9.543e-001	7.858e+013
$\sin i$	0.01	0.031	10	9.918e-001	7.858e+014
$\sin i$	0.1	0.016	2	9.806e-001	7.858e+015

TABLE 6. $A = H_{50}$, $\text{cond}(A) \approx 1.42 \times 10^{74}$

Results of DSM, cont.

u_i	δ	cputime (s)	iterations	Error $\frac{\|\|u_{est}-u_{exact}\|\|}{\|\|u_{exact}\|\|}$	Error $\frac{\|\|A^{-1}f_\delta-u\|\|}{\|\|u\|\|}$
1	0.001	0.047	12	4.342e-002	5.055e+013
1	0.01	0.016	11	3.678e-001	5.055e+014
1	0.1	0.016	10	2.653e+000	5.055e+015
i	0.001	0.016	17	5.190e-002	4.219e+012
i	0.01	0.016	15	7.635e-002	4.219e+013
i	0.1	0.016	13	2.452e-001	4.219e+014
$\sin i$	0.001	0.016	11	8.934e-001	7.044e+013
$\sin i$	0.01	0.016	10	1.014e+000	7.044e+014
$\sin i$	0.1	0.000	3	9.518e-001	7.044e+015

TABLE 7. $A = H_{20}$, $\text{cond}(A) \approx 2.45 \times 10^{28}$

u_i	δ	cputime (s)	iterations	Error $\frac{\|\|u_{est}-u_{exact}\|\|}{\|\|u_{exact}\|\|}$	Error $\frac{\|\|A^{-1}f_\delta-u\|\|}{\|\|u\|\|}$
1	0.001	0.000	11	1.847e-002	7.233e+012
1	0.01	0.016	10	7.953e-002	7.233e+013
1	0.1	0.016	9	4.654e-001	7.233e+014
i	0.001	0.016	15	5.355e-002	7.955e+011
i	0.01	0.000	14	6.534e-002	7.955e+012
i	0.1	0.016	13	1.139e-001	7.955e+013
$\sin i$	0.001	0.016	11	8.366e-001	9.986e+012
$\sin i$	0.01	0.000	10	8.754e-001	9.986e+013
$\sin i$	0.1	0.000	2	9.428e-001	9.986e+014

TABLE 8. $A = H_{15}$, $\text{cond}(A) \approx 6.12 \times 10^{20}$

4.3. **Results of Regularization.**

u_i	δ	cputime (s)	Error $\frac{\|\|u_{est}-u_{exact}\|\|}{\|\|u_{exact}\|\|}$	Error $\frac{\|\|A^{-1}f_\delta-u_{exact}\|\|}{\|\|u_{exact}\|\|}$
1	0.001	0.000	3.157e-002	8.884e+013
1	0.01	0.000	7.057e-002	8.884e+014
1	0.1	0.000	1.661e-001	8.884e+015
i	0.001	0.000	3.519e-002	1.527e+012
i	0.01	0.000	6.799e-002	1.527e+013
i	0.1	0.000	1.218e-001	1.527e+014
$\sin i$	0.001	0.000	9.851e-001	1.253e+014
$\sin i$	0.01	0.000	9.895e-001	1.253e+015
$\sin i$	0.1	0.000	9.910e-001	1.253e+016

TABLE 9. $A = H_{100}$, $\text{cond}(A) \approx 8.60 \times 10^{150}$

u_i	δ	cputime (s)	Error $\frac{\|\|u_{est}-u_{exact}\|\|}{\|\|u_{exact}\|\|}$	Error $\frac{\|\|A^{-1}f_\delta-u_{exact}\|\|}{\|\|u_{exact}\|\|}$
1	0.001	0.000	4.086e-002	5.570e+013
1	0.01	0.000	8.573e-002	5.570e+014
1	0.1	0.000	1.802e-001	5.570e+015
i	0.001	0.000	4.590e-002	1.901e+012
i	0.01	0.000	9.340e-002	1.901e+013
i	0.1	0.000	1.763e-001	1.901e+014
$\sin i$	0.001	0.000	9.654e-001	7.858e+013
$\sin i$	0.01	0.000	9.787e-001	7.858e+014
$\sin i$	0.1	0.000	9.810e-001	7.858e+015

TABLE 10. $A = H_{50}$, $\text{cond}(A) \approx 1.42 \times 10^{74}$

Results of Regularization, cont.

u_i	δ	cputime (s)	Error $\frac{\|\|u_{est}-u_{exact}\|\|}{\|\|u_{exact}\|\|}$	Error $\frac{\|\|A^{-1}f_\delta-u_{exact}\|\|}{\|\|u_{exact}\|\|}$
1	0.001	0.000	3.901e-002	5.055e+013
1	0.01	0.000	8.321e-002	5.055e+014
1	0.1	0.000	1.968e-001	5.055e+015
i	0.001	0.016	5.807e-002	4.219e+012
i	0.01	0.000	1.277e-001	4.219e+013
i	0.1	0.000	2.415e-001	4.219e+014
$\sin i$	0.001	0.000	9.256e-001	7.044e+013
$\sin i$	0.01	0.000	9.474e-001	7.044e+014
$\sin i$	0.1	0.000	9.530e-001	7.044e+015

TABLE 11. $A = H_{20}$, $\text{cond}(A) \approx 2.45 \times 10^{28}$

u_i	δ	cputime (s)	Error $\frac{\|\|u_{est}-u_{exact}\|\|}{\|\|u_{exact}\|\|}$	Error $\frac{\|\|A^{-1}f_\delta-u_{exact}\|\|}{\|\|u_{exact}\|\|}$
1	0.001	0.000	5.105e-002	7.233e+012
1	0.01	0.000	9.887e-002	7.233e+013
1	0.1	0.000	2.192e-001	7.233e+014
i	0.001	0.000	8.427e-002	7.955e+011
i	0.01	0.000	1.248e-001	7.955e+012
i	0.1	0.000	2.637e-001	7.955e+013
$\sin i$	0.001	0.000	8.479e-001	9.986e+012
$\sin i$	0.01	0.000	9.325e-001	9.986e+013
$\sin i$	0.1	0.000	9.472e-001	9.986e+014

TABLE 12. $A = H_{15}$, $\text{cond}(A) \approx 6.12 \times 10^{20}$

4.4. **Error Comparison: DSM & VR, Error.**

N	u_i	δ	**E(DSM)** $\frac{\|\|u_{est}-u_{exact}\|\|}{\|\|u_{exact}\|\|}$	**E(VR)** $\frac{\|\|u_{est}-u_{exact}\|\|}{\|\|u_{exact}\|\|}$	**Difference E(DSM) - E(VR)**
100	1	.001	0.03011	0.03157	-1.460e-003
100	1	.01	0.17020	0.07057	9.963e-002
100	1	.1	1.11200	0.16610	9.459e-001
100	i	.001	0.03448	0.03519	-7.100e-004
100	i	.01	0.06660	0.06799	-1.390e-003
100	i	.1	0.10890	0.12180	-1.29e-002
100	$\sin i$	.001	0.97660	0.98510	-8.500e-003
100	$\sin i$	.01	0.99860	0.98950	9.100e-003
100	$\sin i$	.1	0.99060	0.99100	-4.000e-004
50	1	.001	0.03439	0.04086	-6.470e-003
50	1	.01	0.22200	0.08573	1.363e-001
50	1	.1	1.52600	0.18020	1.346e-000
50	i	.001	0.04558	0.04590	-3.200e-004
50	i	.01	0.08658	0.09340	-6.820e-003
50	i	.1	0.11240	0.17630	-6.390e-002
50	$\sin i$	.001	0.95430	0.96540	-1.110e-002
50	$\sin i$	.01	0.99180	0.97870	1.310e-002
50	$\sin i$	.1	0.98060	0.98100	-4.000e-004
20	1	.001	0.04342	0.03901	4.410e-003
20	1	.01	0.36780	0.08321	2.846e-001
20	1	.1	2.65300	0.19680	2.456e-000
20	i	.001	0.05190	0.05807	-6.170e-003
20	i	.01	0.07635	0.12770	-5.135e-002
20	i	.1	0.24520	0.24150	3.700e-003
20	$\sin i$	.001	0.89340	0.92560	-3.220e-002
20	$\sin i$	.01	1.01400	0.94740	6.660e-002
20	$\sin i$	.1	0.95180	0.95300	-1.200e-003
15	1	.001	0.01847	0.05105	-3.258e-002
15	1	.01	0.07953	0.09887	-1.934e-002
15	1	.1	0.46540	0.21920	2.462e-001
15	i	.001	0.05355	0.08427	-3.072e-002
15	i	.01	0.06534	0.12480	-5.946e-002
15	i	.1	0.11390	0.26370	-1.498e-001
15	$\sin i$	.001	0.83660	0.84790	-1.130e-002
15	$\sin i$	.01	0.87540	0.93250	-5.710e-002
15	$\sin i$	.1	0.94280	0.94720	-4.400e-003

4.5. **Error Comparison: UBM & VR, Error.**

N	u_i	δ	E(UBM) $\frac{\|\|u_{est}-u_{exact}\|\|}{\|\|u_{exact}\|\|}$	E(VR) $\frac{\|\|u_{est}-u_{exact}\|\|}{\|\|u_{exact}\|\|}$	Difference E(UBM) - E(VR)
100	1	.001	0.5320	0.03157	5.0043e-001
100	1	.01	0.6647	0.07057	5.9413e-001
100	1	.1	0.7645	0.16610	5.9840e-001
100	i	.001	0.8002	0.03519	7.6501e-001
100	i	.01	0.8719	0.06799	8.0391e-001
100	i	.1	0.9161	0.12180	7.9430e-001
100	$\sin i$	.001	0.9933	0.98510	8.2000e-003
100	$\sin i$	.01	0.9954	0.98950	5.9000e-003
100	$\sin i$	.1	0.9968	0.99100	5.8000e-003
50	1	.001	0.5353	0.04086	4.9444e-001
50	1	.01	0.6634	0.08573	5.7767e-001
50	1	.1	0.7625	0.18020	5.8230e-001
50	i	.001	0.7994	0.04590	7.5350e-001
50	i	.01	0.8716	0.09340	7.7820e-001
50	i	.1	0.9151	0.17630	7.3880e-001
50	$\sin i$	.001	0.9867	0.96540	2.1300e-002
50	$\sin i$	.01	0.9909	0.97870	1.2200e-002
50	$\sin i$	.1	0.9938	0.98100	1.2800e-002
20	1	.001	0.5376	0.03901	4.9859e-001
20	1	.01	0.6569	0.08321	5.7369e-001
20	1	.1	0.7598	0.19680	5.6300e-001
20	i	.001	0.8006	0.05807	7.4253e-001
20	i	.01	0.8682	0.12770	7.4050e-001
20	i	.1	0.9108	0.24150	6.6930e-001
20	$\sin i$	.001	0.9688	0.92560	4.3200e-002
20	$\sin i$	.01	0.9770	0.94740	2.9600e-002
20	$\sin i$	.1	0.9835	0.95300	3.0500e-002
15	1	.001	0.5355	0.05105	4.8445e-001
15	1	.01	0.6532	0.09887	5.5433e-001
15	1	.1	0.7568	0.21920	5.3760e-001
15	i	.001	0.8005	0.08427	7.1623e-001
15	i	.01	0.8652	0.12480	7.4040e-001
15	i	.1	0.9080	0.26370	6.4430e-001
15	$\sin i$	.001	0.9552	0.84790	1.0730e-001
15	$\sin i$	.01	0.9644	0.93250	3.1900e-002
15	$\sin i$	.1	0.9733	0.94720	2.6100e-002

4.6. **Error Comparison: UBM & VR, CPU Time.**

N	u_i	δ	**UBM** cputime (s)	**VR** cputime (s)	**Difference UBM - VR**
100	1	.001	0.078	0.000	7.8e-002
100	1	.01	0.016	0.000	1.6e-002
100	1	.1	0.031	0.000	3.1e-002
100	i	.001	0.031	0.000	3.1e-002
100	i	.01	0.031	0.000	3.1e-002
100	i	.1	0.031	0.000	3.1e-002
100	$\sin i$	.001	0.047	0.000	4.7e-002
100	$\sin i$	.01	0.031	0.000	3.1e-002
100	$\sin i$	.1	0.031	0.000	3.1e-002
50	1	.001	0.000	0.000	0.0e-002
50	1	.01	0.016	0.000	1.6e-002
50	1	.1	0.031	0.000	3.1e-002
50	i	.001	0.000	0.000	0.0e-002
50	i	.01	0.031	0.000	3.1e-002
50	i	.1	0.031	0.000	3.1e-002
50	$\sin i$	.001	0.031	0.000	3.1e-002
50	$\sin i$	.01	0.031	0.000	3.1e-002
50	$\sin i$	.1	0.031	0.000	3.1e-002
20	1	.001	0.031	0.000	3.1e-002
20	1	.01	0.016	0.000	1.6e-002
20	1	.1	0.031	0.000	3.1e-002
20	i	.001	0.031	0.016	1.5e-002
20	i	.01	0.031	0.000	3.1e-002
20	i	.1	0.000	0.000	0.0e-002
20	$\sin i$	.001	0.000	0.000	0.0e-002
20	$\sin i$	.01	0.031	0.000	3.1e-002
20	$\sin i$	.1	0.031	0.000	3.1e-002
15	1	.001	0.031	0.000	3.1e-002
15	1	.01	0.000	0.000	0.0e-002
15	1	.1	0.016	0.000	1.6e-002
15	i	.001	0.031	0.000	3.1e-002
15	i	.01	0.031	0.000	3.1e-002
15	i	.1	0.031	0.000	3.1e-002
15	$\sin i$	.001	0.000	0.000	0.0e-002
15	$\sin i$	.01	0.000	0.000	0.0e-002
15	$\sin i$	.1	0.016	0.000	1.6e-002

4.7. **Error Comparison: DSM & VR, CPU Time.**

N	u_i	δ	**DSM** cputime (s)	**VR** cputime (s)	**Difference DSM - VR**
100	1	.001	0.250	0.000	2.5000e-001
100	1	.01	0.219	0.000	2.1900e-001
100	1	.1	0.313	0.000	3.1300e-001
100	i	.001	1.750	0.000	1.7500e+000
100	i	.01	1.547	0.000	1.5470e+000
100	i	.1	1.359	0.000	1.3590e+000
100	$\sin i$	.001	0.250	0.000	2.5000e-001
100	$\sin i$	.01	0.219	0.000	2.1900e-001
100	$\sin i$	.1	0.031	0.000	3.1000e-002
50	1	.001	0.125	0.000	1.2500e-001
50	1	.01	0.063	0.000	6.3000e-002
50	1	.1	0.063	0.000	6.3000e-002
50	i	.001	0.109	0.000	1.0900e-001
50	i	.01	0.078	0.000	7.8000e-002
50	i	.1	0.063	0.000	6.3000e-002
50	$\sin i$	.001	0.047	0.000	4.7000e-002
50	$\sin i$	.01	0.031	0.000	3.1000e-002
50	$\sin i$	.1	0.016	0.000	1.6000e-002
20	1	.001	0.047	0.000	4.7000e-002
20	1	.01	0.016	0.000	1.6000e-002
20	1	.1	0.016	0.000	1.6000e-002
20	i	.001	0.016	0.016	0.0000e+000
20	i	.01	0.016	0.000	1.6000e-002
20	i	.1	0.016	0.000	1.6000e-002
20	$\sin i$	.001	0.016	0.000	1.6000e-002
20	$\sin i$	.01	0.016	0.000	1.6000e-002
20	$\sin i$	.1	0.000	0.000	0.0000e+000
15	1	.001	0.000	0.000	0.0000e+000
15	1	.01	0.016	0.000	1.6000e-002
15	1	.1	0.016	0.000	1.6000e-002
15	i	.001	0.016	0.000	1.6000e-002
15	i	.01	0.000	0.000	0.0000e+000
15	i	.1	0.016	0.000	1.6000e-002
15	$\sin i$	.001	0.016	0.000	1.6000e-002
15	$\sin i$	.01	0.000	0.000	0.0000e+000
15	$\sin i$	.1	0.000	0.000	0.0000e+000

5. Conclusion

5.1. **DSM and VR.**

The Dynamical Systems Method consistently required more computation time throughout the experiment, though in cases when $A = H_n$ such that $n < 100$, this extra computation time is very minute, especially when $n = 20$ and $n = 15$ at which time the difference in computation time was often below the accuracy of the software. This extra time is magnified, however, as n gets larger: in the experiment, when $A = H_{100}$ the times for DSM become greater than 1 second in some cases.

Throughout most of the experiments, the accuracy of DSM was close to or better than the accuracy of VR. When accuracy is increased, the residual of error (Error(DSM) - Error(VR)) ranged from $-.03\%$ to -15%, while when accuracy is decreased the residual ranged from .4% to 250%.

Further research on this method is needed to determine the effect a decreased $e_{allowed}$ would have on the accuracy for specific values of u, and perhaps a dynamic allowed error depending on some property of u.

5.2. **UBM and VR.**

The Upper Bound Method required more computation time throughout the experiments, and resulted in decreased accuracy. The residual of accuracy for UBM vs VR was greatest when the elements of u were linear ($u_i = 1, u_i = i$), but least when the elements were periodic ($u_i = \sin i$).

The computation time, however, was very low: almost always less than 1/10th of a second which suggests that further research in lowering the upper bound of the error would be beneficial, though this method is limited to those cases when $||u||$ is known prior to the experiment.

5.3. **UBM and DSM.**

In nearly all cases, the accuracy of DSM was better than with UBM. UBM required much less computation time for large n, though required slightly more computation for small n.

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