THE GAUSS-BONNET THEOREM

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1. Introduction

First of all I would like to thank Professor David Auckly for the hours of his time that he spent teaching me all of the things contained in this paper. Next I would like to thank the NSF for providing me with this opportunity to learn and have fun over the summer.

This paper will give some background of basic differential geometry and topology before proving out a simplified version of the Gauss-Bonnet Theorem.

2. Background

Given a Topological space $\mathbf{X}$, there is a desire to create some kind of coordinate system on $\mathbf{X}$ which will allow us to specify certain points, curves, or any other objects which we wish to study. A simple example of this is the x-y axis in $\mathbb{R}^2$, this coordinate system is especially useful because we can visualize it and easily understand what is meant by parallel or perpendicular, this is not possible though, with more complicated spaces, such as the surface of a sphere or a donut. Fortunately certain spaces can locally be thought of as $\mathbb{R}^n$ and its coordinate system with which we are familiar. This is not true for all spaces, but only a certain class, manifolds. Before defining a n-dimensional manifold, we will first give a few other necessary definitions.

Definition 2.1. A topological space $\mathbf{X}$ is a set with a topology τ. The topology must contain the space itself, as well as the empty set. Any union of sets in the topology must also be in the topology, and a finite intersection of sets from the topology must be contained in the topology.
Example 2.1 : The real line, $\mathbb{R}$ has a topology of all unions of sets of the form, (a,b), a,b $\in \mathbb{R}$, a $\leq$ b.

Definition 2.2. A topological space $\mathbf{X}$ is Hausdorff if for all x,y $\in \mathbf{X}$, there exist open sets U_x, U_y with x $\in U_x$, y $\in U_y$ such that $U_x \bigcap U_y = \emptyset$.
Example 2.2 : The real line, $\mathbb{R}$ is Hausdorff because for all x,y $\in \mathbb{R}$, with x $\neq$ y, $\|$x-y$\| = \delta > 0$. Let U_y=(y-$\delta/2$,y+$\delta/2$), U_x=(x-$\delta/2$,x+$\delta/2$). Clearly x $\in U_x$, y $\in U_y$ and $U_x \bigcap U_y = \emptyset$.

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Definition 2.3. A function F from a topological space $\mathbf{X}$ to a topological space $\mathbf{Y}$ is a homeomorphism if F is onto, one to one, and the inverse function F^{-1} is continuous.

Definition 2.4. A collection of open sets $\{S\}_\alpha$ is a open covering of a space $\mathbf{X}$ if

$$\bigcup_{x\in\alpha} S_x = \mathbf{X}$$

We are now ready for the definition of an n-dimensional manifold. Our definition is taken from Singer's lecture notes on elementary topology, page 97.

Definition 2.5. A smooth differentiable manifold of dimension n is a pair ($\mathbf{X}$,$\mathbf{\Phi}$), where $\mathbf{X}$ is a Hausdorff topological space and $\mathbf{\Phi}$ is a collection of maps such that the following conditions hold:

- **(0):** $\mathbf{X}$ has a countable dense subset ($\mathbf{X}$ is separable).
- **(1):** $\{$ domain φ $\}_{\varphi\in\mathbf{\Phi}}$ is an open covering of $\mathbf{X}$,
- **(2):** each $\varphi \in \mathbf{\Phi}$ maps its domain homeomorphically onto an open set in $\mathbb{R}^n$,
- **(3):** for each φ, $\psi \in \mathbf{\Phi}$ with (domain φ) $\bigcap$ (domain ψ) $\neq \emptyset$, the map $\psi \circ \varphi^{-1}$ is a smooth map from φ(domain $\varphi \bigcap$ domain ψ) $\subset \mathbb{R}^n$ into $\mathbb{R}^n$,
- **(4):** $\mathbf{\Phi}$ is maximal relative to (2) and (3); that is, if ψ is any homeomorphism mapping an open set in $\mathbf{X}$ onto an open set in $\mathbb{R}^n$ such that, for each $\varphi \in \mathbf{\Phi}$ with domain $\varphi \bigcap$ (domain ψ) $\neq \emptyset$, $\psi \circ \varphi^{-1}$ and $\varphi \circ \psi^{-1}$ are smooth maps from φ(domain $\varphi \bigcap$ domain ψ) and ψ(domain $\varphi \bigcap$ domain ψ) into $\mathbb{R}^n$ - then $\psi \in \mathbf{\Phi}$.

The nice thing about n-manifolds is that they can be thought of, locally as $\mathbb{R}^n$. For example, the sphere S^2 when looked at "closely" looks like $\mathbb{R}^2$. We can describe what happens on S^2 by describing what happens in $\mathbb{R}^2$ with appropriate maps between the two spaces. Another interesting example is the torus, T^2. I will now show how this 2-dimensional manifold is easily described relative to $\mathbb{R}^2$. We are used to working in $\mathbb{R}^3$ with the x, y, z coordinate system, with i being a unit vector in the $x-$direction, j a unit vector in the $y-$direction, and k a unit vector in the $z-$direction. Some other common labels for these are e_1 for i, e_2 for j, and e_3 for k. In general, in $\mathbb{R}^n$ we may think of e_n as a unit vector in the x^n direction. Another common notation is ∂x^n for a unit vector in the x^n direction. A infinitesimal displacement in the x^j direction is denoted as dx^j, and this also denotes a function which, when applied to a vector, will "pull out" the jth coordinate.
Example 2.3 : In $\mathbb{R}^n$, $dx^k(ae_1 + be_2 + ... + ce_k + ... + de_n) = dx^k(a\partial x^1 + b\partial x^2 + ... + c\partial x^k + ... + d\partial x^n) = c$.

Definition 2.6. The tensor product, $\otimes$ works in the following way. Given two functions $f(x_1, x_2, ..., x_k)$ and $g(x_1, x_2, ..., x_m)$;
$(f \otimes g)(x_1, x_2, ..., x_{k+m}) = f(x_1, x_2, ..., x_k)g(x_{k+1}, x_{k+2}, ..., x_{k+m})$.

Definition 2.7. The wedge product, $\wedge$ is related to the tensor product, in fact it is defined as; $du \wedge dv := du \otimes dv - dv \otimes du$. Clearly $dv \wedge du = -du \wedge dv$ *and* $dv \otimes (du \otimes dt) = dv \otimes du \otimes dt = (dv \otimes du) \otimes dt$.
Example 2.4 :$du \otimes dv \otimes dt(a\partial_u + b^2\partial_v + c\partial_t) = ab^2c$.

Definition 2.8. In $\mathbb{R}^n$ the inner product is given by $g = dx_1 \otimes dx_1 + dx_2 \otimes dx_2 + ... + dx_n \otimes dx_n$.

Definition 2.9. In $\mathbb{R}^n$ a infinitesmal element of volume is defined as $\mathrm{d}_{vol} = dx_1 \wedge dx_2 \wedge ... \wedge dx_n$.

Now we can find a homeomorphism from open subsets of $\mathbb{R}^2$ to dense open subsets of T^2 where we think of T^2 as a subset of $\mathbb{R}^3$, having a outer radius L and inner radius D. Let H: $\mathbb{R}^2 \longrightarrow \mathbb{R}^3$ where $H = (H_x,\, H_y,\, H_z)$,

$$H_x = (L + D\cos(u))\cos(v) = L\cos(v) + D\cos(u))\cos(v)$$

$$H_y = (L + D\cos(u))\sin(v) = L\sin(v) + D\cos(u))\sin(v)$$

$$H_z = D\sin(u)$$

The metric g_T on the torus can now easily be determined;

$$dH_x = -L\sin(v)dv - D\sin(u)\cos(v)du - D\cos(u)\sin(v)dv$$

$$dH_x = L\cos(v)dv - D\sin(u)\cos(v)du + D\cos(u)\cos(v)dv$$

$$dH_z = D\cos(u)du$$

$$g_T = dH_x \otimes dH_x + dH_y \otimes dH_y + dH_z \otimes dH_z$$

A simple computation gives us;

$$d_{vol} = LD + D^2\cos(u)du \wedge dv$$

$$g_T = (L^2 + 2LD(\cos(u))^2 du \otimes du + D^2 dv \otimes dv$$

We know ∂_u and ∂_v are a orthonormal basis for $\mathbb{R}^2$. Because H is a homeomorphism, it does not take ∂_u and ∂_v to parallel vectors in $\mathbb{R}^3$. We guess that (∂_u,∂_v) is an orthogonal basis on T^2, we check that g(∂_u,∂_v) = 0, so the two vectors are orthogonal, but g(d_u,d_u)=D^2, g(∂_v,∂_v)=(L+Dcos(u))^2, therefore, in order to have a local orthonormal basis we set,

$$e_u = \frac{\partial_u}{D}$$

$$e_v = \frac{\partial_v}{L + D\cos(u)}$$

We can now define θ^1 and θ^2 such that, when applied to r = a∂_u+b∂_v give the following results:

$$\theta^1(r) = a$$

$$\theta^2(r) = b$$

After inspection, we find that;

$$\theta^1 = Ddu$$

$$\theta^2 = (L + D\cos(u))dv$$

We can now perform calculations on T^2 by working in tangent spaces of the much more familiar form, $\mathbb{R}^2$ with the use of $e_u, e_v, \theta^1, \theta^2, d_{vol}$, and g_T.

3. Gauss-Bonnet Theorem

Now that we can think of manifolds in terms of $\mathbb{R}^n$, we ask what parallel means in different spaces. We will now restrict our attention to smooth 2-manifolds. In $\mathbb{R}^2$ parallel lines never intersect, translate a vector any way you want, keeping it parallel to its initial arrangement and it will always point in the same direction. This property does not hold in general, for example on a sphere, a vector may start pointing one way, move around while remaining parallel, and return to its initial position but point in a different direction. We start with a manifold $\mathbf{M}$, and begin to think of a way to describe parallel transport on this manifold. Given a vector $v_0 \in \mathbf{M}$. Let $v(t)$ be a family of parallel vectors as v moves from $v(0) = v_0$ around a closed curve Γ to $v(1)$.

$$v(t) = v^1(t)e_1 + v^2(t)e_2$$

$$\Gamma(t) = (\varphi(t), \theta(t)), t \in [0, 1]$$

As v moves along Γ it remains parallel and therefore its length should stay preserved. Given the initial vector v_0 we should be able to compute $v(t)$. Let $\mathrm{d}v/\mathrm{dt} = Av$, where A is a matrix dependent on t and Γ For any other vector $w \in \mathbf{M}$, starting at the same position as v, and also moving along Γ while staying parallel, we have $\mathrm{d}w/\mathrm{dt} = Aw$. and $\mathrm{d}/\mathrm{dt}(v \cdot w) = 0$.

$$d/dt(v \cdot w) = 0$$

$$dv/dt \cdot w + v \cdot dw/dt = 0$$

$$Av \cdot w + v \cdot Aw = 0$$

$$Av \cdot w + A^T v \cdot w = 0$$

$$(A + A^T)(v) \cdot w = 0$$

Since this is true for all v, w, we get the result that;

$$(A + A^T)(v) = 0$$

$$A = -A^T$$

$$-A = A^T$$

$$A = \begin{bmatrix} a_1^1 & a_2^1 \\ a_1^2 & a_2^2 \end{bmatrix}$$

Since $-A = A^T$,

$$a_1^1 = a_2^2 = 0$$

$$a_2^1 = -a_1^2$$

$$A = \begin{bmatrix} 0 & a_2^1 \\ -a_2^1 & 0 \end{bmatrix}$$

We can solve the following system of equations to find A,

$$d\theta^i + a_j^i \wedge \theta^j = 0, i, j \in \{1, 2\}$$

Which gives us the following summations:

$$d\theta^1 + a_1^1 \wedge \theta^1 + a_2^1 \wedge \theta^2 = 0$$

$$d\theta^2 + a_1^2 \wedge \theta^1 + a_2^2 \wedge \theta^2 = 0$$

Taking into account that $a_1^1 = a_2^2 = 0$ and $a_2^1 = -a_1^2$ we get;

$$d\theta^1 + a_2^1 \wedge \theta^2 = 0$$

$$d\theta^2 - a_2^1 \wedge \theta^1 = 0$$

And now the following equation can be solved for v(t),

$$dv + A(d\Gamma/dt)v = 0$$

Now we can find another quantity, $\Omega = \mathrm{d}A + A \wedge A$, which gives us the curvature K of the manifold we are working on, by the formula:

$$\Omega = K \cdot d_{vol}.$$

Example 2.5 : As an example, the parallel transport matrix A will be determined for T^2, as well as Ω in order to find curvature. We have already calculated the following for T^2:

$$g_T = (L^2 + 2LD(\cos(u))^2 du \otimes du + D^2 dv \otimes dv$$

$$e_u = \frac{\partial_u}{D}$$

$$e_v = \frac{\partial_v}{L + D\cos(u)}$$

$$\theta^1 = Ddu$$

$$\theta^2 = (L + D\cos(u))dv$$

$$d_{vol} = LD + D^2 \cos(u) du \wedge dv$$

Therefore, evaluating;

$$d\theta^1 + a_2^1 \wedge \Theta^2 = 0$$

$$d\theta^2 - a_2^1 \wedge \Theta^1 = 0$$

gives us:

$$a_2^1 = -a_1^2 = \sin(u)dv, so\ we\ have,$$

$$A = \begin{bmatrix} 0 & \sin(u)dv \\ -\sin(u)dv & 0 \end{bmatrix}$$

Now we have, $\Omega = \mathrm{d}A + A \wedge A$, and $A \wedge A = 0$, giving us,

$$\Omega = dA = \begin{bmatrix} 0 & \cos(u) \\ -\cos(u) & 0 \end{bmatrix} du \wedge dv \ = K \cdot d_{vol}$$

Therefore;

$$K = \frac{cos(u)}{DL + D^2 \cos(u)}$$

We now begin our proof of the Gauss-Bonnet Theorem which states:

$$\sum exterior\ angles = 2\pi - \int_D K \cdot d_{vol_D}$$

Where D is a smooth disk in a 2-dimensional manifold M. First we must set up a orthonormal frame $\{e_1,e_2\}$, defined for every $d \in D$. Now we define $\{v,u\}$,an orthonormal, parallel 2-frame on ∂D. Let ∂D be parametrized by $\Gamma = (x(t), y(t)), t \in [0, 1]$. where $x(0) = x(1)$ and $y(0) = y(1)$. The $\{v,u\}$ parallel 2-frame keeps $u(t)$ and $v(t)$ parallel for all t as they travel around ∂D, in $\mathbb{R}^2$ you would expect $u(0)$ and $v(0)$ to equal $u(1)$ and $v(1)$ respectively, but this is not true for more complicated surfaces such as a sphere or a torus, therefore we want to measure how much u changes as it moves from $u(0)$ to $u(1)$, this is what we call the sum of the exterior angles. We now define T as the unit tangent vector to ∂D, we can write T in several ways, relative to the orthogonal parallel frame,

$$T = \cos(\alpha(t))u(t) + \sin(\alpha(t))v(t)$$

or relative to the given coordinate system,

$$T = \cos(\varphi(t))e_1(x(t), y(t)) + \sin(\varphi(t))e_2(x(t), y(t))$$

Here, $\alpha(t)$ is the angle between the tangent vector T and the parallel vector u, while φ is the angle between the coordinate vector e_1 and T, if we write u in terms of the coordinate frame, we get;

$$u(t) = \cos(\theta(t))e_1(x(t), y(t)) + \sin(\theta(t))e_2(x(t), y(t))$$

Now θ is the angle between the coordinate vector e_1 and u, it becomes clear that,

$$\theta = \varphi - \alpha$$

We are now ready to prove the theorem,

$$\sum exterior\ angles = 2\pi - \int_D K \cdot d_{vol_D}$$

Recall that

$$\sum exterior\ angles$$

is a measure of how much T changes from $T(0)$ to $T(1)$ and is therefore equals the difference of the angle it makes with u at t = 0 and t = 1;

$$\alpha(1) - \alpha(0)$$

Now we need to evaluate,

$$\alpha = \varphi - \theta$$
$$\alpha(1) - \alpha(0) = \varphi(1) - \varphi(0) - \theta(1) + \theta(0)$$
$$\alpha(1) - \alpha(0) = \varphi(1) - \varphi(0) - (\theta(1) - \theta(0))$$

Because $x(0) = x(1)$, $y(0) = y(1)$, we know that $T(0)$=$T(1)$ and $e_1(0) = e_1(1)$, therefore the angle between $T(1)$ and $e_1(1)$, $\varphi(1)$ is the same as $\varphi(0)$, the angle between $T(0)$ and $e_1(0)$. So the Tangent vector must have rotated through $2n\pi$. Since this is true for any continuous deformation of ∂D, it is true for a circle, for which the tangent's rotation is 2π. There is no way to continuosly deform the boundary and have the angle of rotation continuously change from 2π to any other multiple $n\pi$, therefore it must have rotated 2π no matter what ∂D is. Therefore we have the following;

$$\alpha(1) - \alpha(0) = 2\pi - (\theta(1) - \theta(0))$$
$$\theta(1) - \theta(0) = \int_0^1 \theta(t)\, dt$$

Now we recall that;

$$u(t) = \cos(\theta(t))e_1(x(t), y(t)) + \sin(\theta(t))e_2(x(t), y(t))$$

and therefore;

$$\frac{du(t)}{dt} = -\sin(\theta(t))\frac{d\theta}{dt}e_1 + \sin(\theta(t))\frac{d\theta}{dt}e_2$$

Where we have assumed $e1, e2$ to be a unchanging frame.

$$\partial D = (x(t), y(t)), \quad \frac{d\partial D}{dt} = \frac{dx}{dt}\partial x + \frac{dy}{dt}\partial y$$

Recall that as u is parallel transported around ∂D, we have,

$$du + A(\frac{d\partial D}{dt})u = 0$$

Performing the calculation we get;

$$[e_1 \sin(\theta) + e_2 \cos(\theta)][a^1_{21}\frac{dx}{dt} + a^1_{22}\frac{dy}{dt} - \frac{d\theta}{dt}] = 0$$

Because $\|e_1 \sin(\theta) + e_2 \cos(\theta)\| = 1$, we know that,

$$[a^1_{21}\frac{dx}{dt} + a^1_{22}\frac{dy}{dt} - \frac{d\theta}{dt}] = 0$$

so now we have,

$$a^1_{21}\frac{dx}{dt} + a^1_{22}\frac{dy}{dt} = \frac{d\theta}{dt}$$

therefore,

$$\int_0^1 \theta(t)\, dt = \int_0^1 (a^1_{21}\frac{dx}{dt} + a^1_{22}\frac{dy}{dt})\ dt$$
$$\int_0^1 (a^1_{21}\frac{dx}{dt} + a^1_{22}\frac{dy}{dt})\ dt = \int_{\partial D} a^1_{21}dx + a^1_{22}dy = \int_{\partial D} a^1_2$$

$$\int_{\partial D} a_2^1 = \int_D da_2^1 = \int_D \Omega_2^1 = \int_D K d_{vol_D}$$

We have achieved the the Gauss-Bonnet theorem;

$$\theta(1) - \theta(0) = \sum exterior\ angles = 2\pi - \int_D K \cdot d_{vol_D} \qquad \square$$