An Introduction to Partial Differential Equations and a Problem from Composite Materials

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1 Separation of variables

1.1 Outline

The most basic way of solving linear partial differential equations and the first method learned by the vast majority of students is the method of separation of variables. This method consists of the following: We attempt to find a family of solutions with a special form, and then try to obtain all solutions as linear combinations of these special solutions. The special requirement we place on these initial functions is that not only do we ask them to solve the partial differential equation and any homogeneous or periodic boundary conditions, but we also require them to be products of functions which each have only one independent variable.

Our first step in applying this method is to substitute these functions into the original partial differential equation and attempt to use algebra to bring the resulting equation into a form where the independent variables appear only on opposite sides of the equality; hence the name "separation of variables." At this point, assuming that we have been successful in separating the variables (and assuming for simplicity that the original partial differential equation is of second order), our equation should have the form

$$F(x, f(x), f'(x), f''(x)) = G(y, g(y), g'(y), g''(y))$$

where F and G are functions which are determined completely by the original partial differential equation. Now whatever f and g turn out to be, we can also say that our equation now has the form

$$\mathcal{F}(x) = \mathcal{G}(y)$$

where $\mathcal{F}(x) := F(x, f(x), f'(x), f''(x))$ and $\mathcal{G}(y) := G(y, g(y), g'(y), g''(y))$. Next by alternately varying x and y while the other variable is held fixed, it is clear that we must have

$$\mathcal{F}(x) = \mathcal{G}(y) = \text{constant} \tag{1.1}$$

Solving $\mathcal{F}(x) = \mathrm{C}$ and $\mathcal{G}(y) = \mathrm{C}$ amounts to solving an ordinary differential equation. At this point when we introduce any homogeneous or periodic boundary conditions demanded by the original partial differential equation we discover that most values of the constant on the right hand side of equation (1.1) lead us only to the trivial solution. On the other hand, for a wide class of linear boundary value problems which include Laplace's equation on a disk, the heat equation on a line segment, and the wave equation on a line segment, we will obtain a sequence of "good" constants that lead to a corresponding sequence of nontrivial solutions. Due to the special properties enjoyed by the boundary values of this sequence of solutions, it turns out that the solution of the original partial differential equation will always be a linear combination of these functions, and there is a simple way to derive an explicit formula for the coefficients in this expression. We will explore those properties in the next section, but now we turn to the application of the rest of the method to Laplace's equation on a disk.

1.2 Examples

We start by solving Laplace's equation on a disk as an example of the application of separation of variables. (As it turns out, we will need the solutions that we find here later on when we turn to our problem from composite materials.) We seek a function u which satisfies the following boundary value problem:

$$\begin{aligned} \Delta u = 0 \quad &\text{in} \quad D_R \\ u(R, \theta) = f(\theta) \quad &\text{on} \quad \partial D_R \ . \end{aligned} \tag{1.2}$$

The Laplace operator in $\mathbb{R}^2$ is given by

$$\Delta U = U_{xx} + U_{yy} \ .$$

In polar coordinates, by a lengthy but elementary exercise using the multivariable chain rule, we can verify

$$\Delta U = U_{rr} + \frac{1}{r}U_r + \frac{1}{r^2}U_{\theta\theta} \ . \tag{1.3}$$

Thus, Laplace's equation in $\mathbb{R}^2$ for a function $U = U(r,\theta)$ becomes

$$0 = U_{rr} + \frac{1}{r}U_r + \frac{1}{r^2}U_{\theta\theta} \ . \tag{1.4}$$

We assume our solution takes the form $U(r,\theta) = g(r)h(\theta)$ and by plugging this into (1.4) we obtain

$$\begin{aligned} 0 &= g''(r)h(\theta) + \frac{1}{r}g'(r)h(\theta) + \frac{1}{r^2}g(r)h''(\theta) \\ &= (g''(r) + \frac{1}{r}g'(r))h(\theta) + \frac{1}{r^2}g(r)h''(\theta) \end{aligned}$$

or

$$\frac{h''(\theta)}{h(\theta)} = \frac{-g''(r) + \frac{1}{r}g'(r)}{\frac{1}{r^2}g(r)} = -\left(\frac{r^2g''(r) + rg'(r)}{g(r)}\right) \tag{1.5}$$

Since the left hand side is a function of θ and the right hand side is a function of r, the only way a function of θ can equal a function of r is for both functions to be the same constant. This is exactly the situation we had in Equation (1.1). We call this constant δ, and we split the problem into three cases according to whether δ is positive, zero, or negative.

Case 1: $\delta = \lambda^2 > 0$, and so

$$\frac{h''(\theta)}{h(\theta)} = \lambda^2,$$

then,

$$h''(\theta) - h(\theta)\lambda^2 = 0.$$

We solve the ordinary differential equation and we get

$$h(\theta) = Ae^{\lambda\theta} + Be^{-\lambda\theta}$$

where A and B are constants. Then, we need to consider the periodicity condition:

$$U(r,-\pi) = U(r,\pi). \tag{1.6}$$

We can express (1.6) in terms of $g(r)$ and $h(\theta)$

$$g(r)h(-\pi) = g(r)h(\pi)$$

and then either $g(r) \equiv 0$, or

$$\begin{aligned} h(-\pi) &= h(\pi) \\ Ae^{\lambda(-\pi)} + Be^{-\lambda(-\pi)} &= Ae^{\lambda\pi} + Be^{-\lambda\pi} \\ (A-B)e^{\lambda\pi} &= (A-B)e^{-\lambda\pi} \ . \end{aligned} \tag{1.7}$$

Since $g(r) \equiv 0$ gives only the trivial solution, we exclude this possibility, and then since $e^{\lambda\pi} \neq e^{-\lambda\pi}$, we can say $A = B$. Thus, $h(\theta) = A(e^{\lambda\theta} + e^{-\lambda\theta})$. Now we consider the other periodicity condition:

$$U_\theta(r,\pi) = U_\theta(r,-\pi) \tag{1.8}$$

We can express $U_\theta(r,\theta)$ in terms of $g(r)$ and $h(\theta)$

$$\begin{aligned} U_\theta(r,\theta) &= g(r)h'(\theta) \\ &= g(r)\frac{d}{d\theta}(A(e^{\lambda\theta} + e^{-\lambda\theta})) \\ &= g(r)A(\lambda e^{\lambda\theta} - \lambda e^{-\lambda\theta}) \\ &= A\lambda g(r)(e^{\lambda\theta} - e^{-\lambda\theta}) \ . \end{aligned} \tag{1.9}$$

We substitute (1.9) into (1.8) and then after again excluding $g(r) \equiv 0$, we get

$$\begin{aligned} g(r)h'(\pi) &= g(r)h'(-\pi) \\ A\lambda(e^{\lambda\pi} - e^{-\lambda\pi}) &= A\lambda(e^{-\lambda\pi} - e^{\lambda\pi}) \\ 2Ae^{\lambda\pi} &= 2Ae^{-\lambda\pi} \\ Ae^{\lambda\pi} &= Ae^{-\lambda\pi}. \end{aligned}$$

Since $e^{\lambda\pi} \neq e^{-\lambda\pi}$, A must be 0 and then B is also 0, and so we obtain only the trivial solution in this case.

Case 2: $\delta = 0$, and so

$$\frac{h''(\theta)}{h(\theta)} = 0. \tag{1.10}$$

The general solution of (1.9) is

$$h(\theta) = A\theta + B$$

where A and B are constants. Then we consider periodicity conditions. From (1.6), we can derive

$$A(-\pi) + B = A\pi + B. \tag{1.11}$$

Equation (1.10) forces A to be 0. (Note that Equation (1.7) imposes nothing new.) Thus, $U(r,\theta) = Bg(r)$, and $g(r)$ satisfies

$$0 = r^2 g''(r) + rg'(r).$$

Let $G(r) = g'(r)$. Then,

$$\begin{aligned} 0 &= r^2 G'(r) + rG(r) \\ &= rG'(r) + G(r) \\ &= \frac{d}{dr}(rG(r)). \end{aligned}$$

Since its derivative is zero, $rG(r)$ is constant. Thus,

$$G(r) = \frac{\tilde{A}}{r} \quad \text{and} \quad g(r) = \tilde{A}\ln r + \tilde{B}$$

where $\tilde{A}$ and $\tilde{B}$ are constants. Then, boundedness forces $\tilde{A} = 0$. Therefore, the only nontrivial solutions in this case are constants.

Case 3: $\delta = \lambda^2 < 0$, and so

$$\frac{h''(\theta)}{h(\theta)} = -\lambda^2,$$

and

$$h''(\theta) - h(\theta)\lambda^2 = 0.$$

This ordinary differential equation has the general solution

$$h(\theta) = A\cos\lambda\theta + B\sin\lambda\theta.$$

From (1.6), we can derive

$$A\cos(-\lambda\pi) + B\sin(-\lambda\pi) = A\cos(\lambda\pi) + B\sin(\lambda\pi)$$

then,

$$2B\sin(\lambda\pi) = 0$$

Thus, we can say either $B = 0$ or $\lambda = n \in \mathbb{N}$. Next, we need to consider the other periodicity condition. As in (1.8), we can derive

$$U_\theta(r,\theta) = g(r)(-A\lambda\sin\lambda\theta + B\lambda\cos\lambda\theta)$$

By excluding $g(r) \equiv 0$ as before we get

$$-A\lambda\sin(-\lambda\pi) + B\lambda\cos(-\lambda\pi) = -A\lambda\sin(\lambda\pi) + B\lambda(\cos\lambda\pi)$$

and

$$2A\lambda\sin(\lambda\pi) = 0.$$

Therefore, we can say either $A = 0$ or $\lambda = n \in \mathbb{N}$. We note, however, that $\lambda = n$ leads to a nontrivial solution. On the other hand, if $\lambda \neq n$, then it forces both $A = 0$ and $B = 0$, which of course gives us only the trivial solution, so henceforth we are only interested in the case $\lambda = n$.

Now, we need to find $g(r)$. From (1.4), we have

$$-\left(\frac{r^2g''(r) + rg''(r)}{g(r)}\right) = -\lambda^2 = -n^2$$

and

$$r^2g''(r) + rg'(r) - n^2g(r) = 0$$

We solve this Euler Equation and get

$$g(r) = Ar^{-n} + Br^n.$$

Once again, boundedness forces $A = 0$, and since we know $h(\theta)$ and $g(r)$, we have the family of solutions of the partial differential equation

$$U_n(r,\theta) = r^n(A\cos(n\theta) + B\sin(n\theta)). \tag{1.12}$$

Let

$$\begin{aligned} \tilde{V}_n(r,\theta) &= r^n\cos(n\theta) \\ \tilde{W}_n(r,\theta) &= r^n\sin(n\theta) \end{aligned}$$

then,

$$\begin{aligned}\tilde{V_n}(R,\theta) &= R^n\cos(n\theta)\\ \tilde{W_n}(R,\theta) &= R^n\sin(n\theta)\end{aligned}$$

In addition, let

$$\begin{aligned}V_n(r,\theta) &= \left(\frac{r}{R}\right)^n\cos(n\theta) \qquad n=0,\cdots,\infty\\ W_n(r,\theta) &= \left(\frac{r}{R}\right)^n\sin(n\theta) \qquad n=1,\cdots,\infty\end{aligned} \tag{1.13}$$

and this is the desired family of nontrivial solutions. Observe that any function which is a linear combination of the functions in Equation (1.12) is also a linear combination of the functions in Equation (1.13). Note that the boundary values of these solutions are simply

$$\begin{aligned}V_n(R,\theta) &= \cos(n\theta)\\ W_n(R,\theta) &= \sin(n\theta).\end{aligned}$$

(These last two equations show why the family $\{V_n\}\cup\{U_n\}$ is slightly preferable to the family $\{\tilde{V_n}\}\cup\{\tilde{U_n}\}$.)

Now the key questions are

1. Can $f(\theta)$ be expanded as a linear combination of these boundary values?

2. If so, can we compute the coefficients in our trigonometric series?

The answer to these questions is yes as we will see in the next section.

2 Fourier series and orthogonal functions

To expand $f(\theta)$ as a linear combination of the family of trigonometric functions we obtained as boundary values of solutions in the previous section, we will want to understand the idea of orthogonality as it applies to infinite dimensional vector spaces of functions. We assume the reader is familiar with the notion of orthogonality in finite dimensions, and in fact, the formulas we derive can be found by exact analogy with the formulas found in Linear

Algebra for the new coefficients of a vector under a change of orthogonal basis. We turn to this derivation first.

Suppose we have an orthogonal basis $\{e_1, e_2, e_3\}$ of $\mathbb{R}^3$ and a vector $\vec{V} \in \mathbb{R}^3$. Then since we can express $\vec{V}$ as a linear combination of e_1, e_2, and e_3, there exist scalars a_1, a_2, and a_3 such that

$$\vec{V} = a_1 e_1 + a_2 e_2 + a_3 e_3. \tag{2.1}$$

To find these scalars , we will use the dot product. By taking the dot product of both sides of (2.1) with e_j we obtain

$$\begin{aligned} \vec{V} \cdot e_j &= (a_1 e_1 + a_2 e_2 + a_3 e_3) \cdot e_j \\ &= a_1 e_1 \cdot e_j + a_2 e_2 \cdot e_j + a_3 e_3 \cdot e_j \\ &= a_j \parallel e_j \parallel^2 \qquad \text{by orthogonality} \end{aligned}$$

So,

$$a_j = \frac{\vec{V} \cdot e_j}{\parallel e_j \parallel^2}.$$

Now consider the set, $L^2(S^1)$, of periodic functions $f(\theta)$ on $[0, 2\pi]$ which satisfy

$$\int_0^{2\pi} (f(\theta))^2 \, d\theta \ < \infty$$

One can check that $L^2(S^1)$ forms a vector space. Now for any $f, g \in L^2(S^1)$ we define the bilinear functional $f \cdot g = \int_0^{2\pi} f(\theta) g(\theta) d\theta$ and one can check that this functional, "$\cdot$", satisfies the hypotheses required for it to be a dot product.

Next we consider the following set of functions.

$$\{1\} \cup \{\cos(n\theta)\} \cup \{\sin(n\theta)\}$$

These functions are vectors in $L^2(S^1)$. Our motivation for considering this particular set of functions is that this set is exactly what we obtained as boundary values of our special solutions in the previous section by separation of variables for Laplace's equation on a disk. We need to take dot products to determine whether these functions are orthogonal. By direct computation

of the integrals we verify

$$\left.\begin{aligned}
\cos(n\theta)\cdot\cos(m\theta) &= \int_0^{2\pi}\cos(n\theta)\cos(m\theta)\,d\theta &&= 0 \qquad n\neq m\\
\sin(n\theta)\cdot\sin(m\theta) &= \int_0^{2\pi}\sin(n\theta)\sin(m\theta)\,d\theta &&= 0 \qquad n\neq m\\
\sin(n\theta)\cdot\cos(m\theta) &= \int_0^{2\pi}\sin(n\theta)\cos(m\theta)\,d\theta &&= 0
\end{aligned}\right\} \tag{2.2}$$

and so indeed we can say that this set of functions forms an orthogonal set. Now just because we have an infinite orthogonal set, does not mean that our set is "large" enough to be a basis for $L^2(S^1)$. (Consider the set $\{\hat{\mathrm{i}},\hat{\mathrm{j}}\}\neq\{\hat{\mathrm{i}},\hat{\mathrm{j}},\hat{\mathrm{k}}\}$ in $\mathbb{R}^3$ for comparison.) The fact that this set *is* a basis is a consequence of the following fundamental theorem of Fourier analysis.

Theorem 2.1. *Let $f(t)$ be piecewise-differentiable, continuous, and periodic on $[-L,L]$, then there exist constants $\{a_n\}_{n=0}^{\infty}$, $\{b_n\}_{n=1}^{\infty}$ such that*

$$f(t)=\frac{a_0}{2}+\lim_{N\to\infty}\sum_{n=1}^{N}\left(a_n\cos\left(\frac{n\pi t}{L}\right)+b_n\sin\left(\frac{n\pi t}{L}\right)\right). \tag{2.3}$$

Since the set of trigonometric functions considered above forms an orthogonal basis, we can figure out the coefficients a_0, a_n, and b_n exactly as we did in the finite dimensional case. Now instead of talking the dot product (in the usual finite dimensional sense) of both sides of Equation (2.1) with the vectors e_1, e_2, and e_3, we need to take the dot product (in the $L^2(S^1$ sense) of both sides of Equation (2.3) with our orthogonal family of trigonometric functions. Starting with the function 1, we get

$$\begin{aligned}
\int_{-L}^{L} f(t)\cdot 1dt &= \int_{-L}^{L}\frac{a_0}{2}\cdot 1dt+\sum_{n=1}^{\infty}0\\
&= a_0\cdot L
\end{aligned}$$

Thus,

$$a_0=\frac{1}{L}\int_{-L}^{L}f(t)dt. \tag{2.4}$$

Similarly, by taking the dot product of both sides of (2.3) with the function $\cos\left(\frac{m\pi t}{L}\right)$, we get

$$\begin{aligned}
\int_{-L}^{L} f(t)\cdot\cos\left(\frac{m\pi t}{L}\right)dt &= 0+\sum_{n\neq m}^{\infty} 0+\int_{-L}^{L} a_m\cos\left(\frac{m\pi t}{L}\right)\cos\left(\frac{m\pi t}{L}\right)dt \\
&= a_m\int_{-L}^{L}\cos^2\left(\frac{m\pi t}{L}\right)dt \\
&= a_m\cdot L
\end{aligned}$$

Then, after doing the same computation with $\sin\left(\frac{m\pi t}{L}\right)$ and after rearranging, we get

$$\left.\begin{aligned}
a_m &= \frac{1}{L}\int_{-L}^{L} f(t)\cos\left(\frac{m\pi t}{L}\right)dt \qquad m\geq 0 \\
b_m &= \frac{1}{L}\int_{-L}^{L} f(t)\sin\left(\frac{m\pi t}{L}\right)dt \qquad m\geq 1.
\end{aligned}\right\} \tag{2.5}$$

Therefore, $f(\theta)$ can be expanded as a linear combination of the boundary values of the special solutions, and we have explicit formulas for the coefficients in this trigonometric series. However, before we get to our problem, there is one more important concept to understand, and we will explain it in the next section.

3 Weak solutions of partial differential equations

Often scientists produce a partial differential equation as a mathematical model for a physical phenomenon. However, implicit in the fact that they have a partial differential equation for their model is that they expect or need differentiability of the solutions in order to understand their partial differential equation in its "classical sense." Unfortunately, scientists will occasionally have a partial differential equation which does a fantastic job of describing the observed smooth solutions, but where physically there seem to be nonsmooth "solutions" to the same phenomenon or process. Mathematicians have found ways around this difficulty by generalizing the notion of what it means to solve a partial differential equation. We begin with a motivational example.

3.1 The wave equation

Suppose we have a rope which is stretched horizontally. Then when it vibrates, we can see waves traveling along the length of the rope. We can express this phenomenon in terms of the following partial differential equation (known as the wave equation) which can be derived using Newtonian mechanics and a few approximations,

$$V_{xx} = V_{tt}. \tag{3.1}$$

In this equation V represents displacement from the horizontal position. If the initial position or velocity of our rope is not differentiable, then we can observe nondifferentiable waves traveling on our rope when we perform empirical experiments. We can write down a mathematical function as an example of this phenomenon.

$$U_1(x,t) = \begin{cases} 0 & x - t \leq -1 \\ 1 - |x - t| & -1 \leq x - t \leq 1 \\ 0 & 1 \leq x - t. \end{cases} \tag{3.2}$$

Equation (3.1) does not make sense for this case since U_1 is not even once differentiable. Therefore, we need to define a generalized notion of solution to a partial differential equation. One attempt at defining a weak solution might be to demand that the partial differential equation holds wherever the putative solution is sufficiently differentiable, and the example above works with this definition.

The function

$$U_2(x,t) = \begin{cases} 0 & x - 2t \leq -1 \\ 1 - |x - 2t| & -1 \leq x - 2t \leq 1 \\ 0 & 1 \leq x - 2t. \end{cases} \tag{3.3}$$

would also be a weak solution by this definition, however (3.3) is not observed in nature whereas (3.2) *is* observed. So this simple attempt at generalizing the notion of solution fails. In the observed phenomenon the moving triangle must travel with a fixed speed which is not imposed by this definition. In short the attempt as stated above is too broad: Too many functions become weak solutions. In other words there are more mathematical "solutions" with this definition, than there are observed physical "solutions." It turns out that mathematicians have found a very flexible way of defining solutions (the so-called "weak solutions" in the next section) which are typically neither too broad nor too restrictive for the purpose of modeling these physical situations.

3.2 The formal definition of a weak solution to a partial differential equation

We would like to state the formal definition of a weak solution to a partial differential equation. This time, we use the wave equation on a rope as our example, but the idea of how to define "weak solutions" to a partial differential equation is very broad, and it will be clear how to extend this definition to any linear partial differential equation from the case we give here. For the sake of comparison, we start with the classical version of our partial differential equation. We seek a V such that

$$V_{tt} = V_{xx} \quad \text{for all} \quad (x,t) \in \mathbb{R} \times (0, +\infty) \tag{3.4}$$

Now assuming that we have such a V, we let φ be any smooth function which vanishes outside a compact subset of $\mathbb{R} \times (0, +\infty)$, and we observe that the integrals

$$\int_{x\in\mathbb{R}} \int_{t\in\mathbb{R}^+} V_{xx}\varphi \; dtdx \quad \& \quad \int_{x\in\mathbb{R}} \int_{t\in\mathbb{R}^+} V_{tt}\varphi \; dtdx$$

are both well defined and are obviously equal. Integrating by parts twice on each integral we conclude that for any φ as above

$$\int_{x\in\mathbb{R}} \int_{t\in\mathbb{R}^+} \varphi_{xx} V dtdx \quad = \quad \int_{x\in\mathbb{R}} \int_{t\in\mathbb{R}^+} \varphi_{tt} V dtdx \tag{3.5}$$

for all classical solutions of the wave equation. One can also show that any twice continuously differentiable function V which satisfies (3.5) for all φ must necessarily be a classical solution of the wave equation. To show this fact we start with the observation that if our weak solution V is twice differentiable then we can integrate each side of Equation (3.5) by parts twice and rearrange to conclude that such V's satisfy

$$\int_x \int_t \varphi(V_{xx} - V_{tt}) \; dtdx = 0 \; . \tag{3.6}$$

Because V is assumed to be twice continuously differentiable, $V_{xx} - V_{tt}$ is continuous in $\mathbb{R} \times \mathbb{R}^+$. Then suppose there exists $(\tilde{x}, \tilde{t}) \in \mathbb{R} \times \mathbb{R}^+$ such that $V_{xx} - V_{tt}$ is positive at $(\tilde{x}, \tilde{t})$. By continuity, $V_{xx} - V_{tt}$ is positive in some small rectangle $D_0 \subset \mathbb{R} \times \mathbb{R}^+$ containing $(\tilde{x}, \tilde{t})$. Take $\varphi \in C_0^1(D_0)$ with $\varphi \geq 0$ in D_0 and $\varphi > 0$ at the center of D_0. Then

$$\int_{x\in\mathbb{R}} \int_{t\in\mathbb{R}^+} \varphi(V_{xx} - V_{tt}) \; dtdx = \int_{D_0} \varphi(V_{xx} - V_{tt}) \; dtdx \; . \tag{3.7}$$

Since φ and $V_{xx} - V_{tt}$ are positive in D_0, the expression on the right hand side of Equation (3.7) is positive, and this fact contradicts (3.6). Thus, we can say that $V_{xx} - V_{tt}$ is always 0. We summarize with the following theorem.

Proposition 3.1. *Let V be a continuous function in $\mathbb{R} \times (0, +\infty)$. Then V is a classical solution of the wave equation if and only if*

$$\begin{gathered} V \text{ is twice continuously differentiable} \\ \& \\ V \text{ satisfies (3.5) for all } \varphi \in C_0^\infty\ (\mathbb{R} \times \mathbb{R}^+). \end{gathered} \tag{3.8}$$

On the other hand, Equation (3.5) continues to make sense for functions V which are merely integrable, and we can compare the set of functions satisfying (3.5) for all suitable φ with the set of classical solutions of the wave equation. Functions satisfying (3.5) for all such φ are said to be *weak solutions* of the wave equation. In the next section we will show that the set of weak solutions of the wave equation includes the function U_1 defined in Equation (3.5), and is therefore a broader class of functions than the set of classical solutions to the wave equation. Unlike our earlier attempt, however, by this definition, U_2 is *not* a weak solution, which is good, since the wave travels at the wrong speed.

3.3 The wave equation revisited

In this section, we will show U_1 as defined in Equation (3.2) is a weak solution of the wave equation. Thus, we substitute (3.5) into (3.2) and then we need to show that

$$\int_{t=0}^{\infty} \int_{x=t-1}^{t+1} (1 - \mid x - t \mid) \varphi_{xx}\ dxdt = \int_{t=0}^{\infty} \int_{x=t-1}^{t+1} (1 - \mid x - t \mid) \varphi_{tt}\ dxdt. \tag{3.9}$$

Because we can simultaneously show that U_1 is a weak solution while showing that U_2 is not, we will define

$$U_k(x,t) = \begin{cases} 0 & x - kt \leq -1 \\ 1 - |x - kt| & -1 \leq x - kt \leq 1 \\ 0 & 1 \leq x - kt. \end{cases} \tag{3.10}$$

and in fact show that U_k is a weak solution if and only if $\mid k \mid = 1$. Let

$$J_1 = \int_{t=0}^{\infty} \int_{x=kt-1}^{kt+1} (1 - \mid x - kt \mid) \varphi_{xx} \; dxdt \tag{3.11}$$

$$J_2 = \int_{t=0}^{\infty} \int_{x=kt-1}^{kt+1} (1 - \mid x - kt \mid) \varphi_{tt} \; dxdt. \tag{3.12}$$

We compute J_1. Let

$$I_1 - I_2 := \int_{t=0}^{\infty} \int_{x=kt}^{kt+1} (kt - x) \varphi_{xx} \; dxdt \tag{3.13}$$

$$I_3 - I_4 := \int_{t=0}^{\infty} \int_{x=kt-1}^{kt} (x - kt) \varphi_{xx} \; dxdt \tag{3.14}$$

and

$$I_5 := \int_{t=0}^{\infty} \int_{x=kt-1}^{kt+1} \varphi_{xx} \; dxdt. \tag{3.15}$$

Then, $J_1 = (I_1 - I_2) + (I_3 - I_4) + I_5$. We simplify (3.13).

$$\begin{aligned} I_1 &= \int_{t=0}^{\infty} kt \int_{x=kt}^{kt+1} \varphi_{xx} \; dxdt \\ &= \int_{t=0}^{\infty} kt \; (\varphi_x(kt+1, t) - \varphi_x(kt, t)) \; dt \end{aligned} \tag{3.16}$$

and

$$I_2 = \int_{t=0}^{\infty} \int_{x=kt}^{kt+1} x \; \varphi_{xx} \; dxdt$$

which, by integration by parts, becomes

$$\begin{aligned} I_2 &= \int_{t=0}^{\infty} \left[(kt+1) \; \varphi_x(kt+1, t) - kt \; \varphi_x(kt, t) - \left(\int_{t=0}^{\infty} \int_{x=kt}^{kt+1} \varphi_x \; dx \right) \right] dt \\ &= \int_{t=0}^{\infty} \left[(kt+1) \; \varphi_x(kt+1, t) - kt \; \varphi_x(kt, t) - \varphi(kt+1, t) + \varphi(kt, t) \right] dt. \end{aligned}$$

So,

$$I_1 - I_2 = \int_{t=0}^{\infty} \left(-\varphi_x(kt+1, t) + \varphi(kt+1, t) - \varphi(kt, t) \right) \; dt. \tag{3.17}$$

Next, we compute (3.14).

$$I_3 = \int_{t=0}^{\infty} \int_{x=kt-1}^{kt} x \ \varphi_{xx} \ dxdt$$

then by integration by parts, we get

$$\begin{aligned} I_3 &= \int_{t=0}^{\infty} \left[(kt\varphi_x(kt,t) - (kt-1)\varphi_x(kt-1,t)) - \int_{x=kt-1}^{kt} \varphi_x(x,t) \ dx \right] dt \\ &= \int_{t=0}^{\infty} \left[kt\varphi_x(kt,t) - (kt-1)\varphi_x(kt-1,t) - \varphi(kt,t) + \varphi(k+1,t)) \right] \ dt. \end{aligned}$$

Then,

$$\begin{aligned} I_4 &= \int_{t=0}^{\infty} kt \int_{x=kt-1}^{kt} \varphi_{xx} \ dxdt \\ &= \int_{t=0}^{\infty} kt \left[(\varphi_x(kt,t) - \varphi_x(kt-1,t)) \right] \ dt \end{aligned}$$

Thus,

$$I_3 - I_4 = \int_{t=0}^{\infty} \left[\varphi_x(kt-1,t) - \varphi(kt,t) + \varphi(kt-1,t) \right] \ dt$$

Also, we compute (3.15).

$$\begin{aligned} I_5 &= \int_{t=0}^{\infty} \int_{x=kt-1}^{kt+1} \varphi_{xx} \ dxdt \\ &= \int_{t=0}^{\infty} \left[\varphi_x(kt+1,t) - \varphi_x(kt-1,t) \right] \ dt \end{aligned}$$

Therefore,

$$J_1 = \int_{t=0}^{\infty} \left[\varphi(kt+1,t) - 2\varphi(kt,t) + \varphi(kt-1,t) \right] \ dt. \tag{3.18}$$

Next, we compute J_2 using Equation (3.12). Applying Fubini's Theorem we get

$$J_2 = \int_{x=-\infty}^{\infty} \int_{t=\frac{x-1}{k}}^{\frac{x+1}{k}} (1 - \mid x - k \mid) \varphi_{tt} \ dtdx$$

Let

$$H_1 - H_2 := \int_{x=-\infty}^{\infty} \int_{t=\frac{x}{k}}^{t=\frac{x+1}{k}} (x - kt)\varphi_{tt}\, dtdx \tag{3.19}$$

$$H_3 - H_4 := \int_{x=-\infty}^{\infty} \int_{t=\frac{x-1}{k}}^{\frac{x}{k}} (kt - x)\varphi_{tt}\, dtdx \tag{3.20}$$

and

$$H_5 := \int_{x=-\infty}^{\infty} \int_{t=\frac{x-1}{k}}^{\frac{x+1}{k}} \varphi_{tt}\, dtdx. \tag{3.21}$$

So $J_2 = (H_1 - H_2) + (H_3 - H_4) + H_5$. We compute (3.19).

$$\begin{aligned} H_1 &= \int_{x=-\infty}^{\infty} x \int_{\frac{x}{k}}^{t=\frac{x+1}{k}} \varphi_{tt}\, dtdx \\ &= \int_{x=-\infty}^{\infty} x \left[\varphi_t \left(x, \frac{x+1}{k} \right) - \varphi_t \left(x, \frac{x}{k} \right) \right] dx, \end{aligned}$$

and

$$H_2 = \int_{x=-\infty}^{\infty} \int_{\frac{x}{k}}^{t=\frac{x+1}{k}} kt\ \varphi_{tt}\, dtdx,$$

so by integration by parts, we get

$$\begin{aligned} H_2 &= \int_{x=-\infty}^{\infty} \left[(x+1)\varphi_t \left(x, \frac{x+1}{k} \right) - x\varphi_t \left(x, \frac{x}{k} \right) - \int_{\frac{x}{k}}^{t=\frac{x+1}{k}} k\varphi_t\, dt \right] dx \\ &= \int_{x=-\infty}^{\infty} \left[(x+1)\varphi_t \left(x, \frac{x+1}{k} \right) - x\varphi_t \left(x, \frac{x}{k} \right) \right. \\ &\qquad \left. -k\varphi \left(x, \frac{x+1}{k} \right) + k\varphi \left(x, \frac{x}{k} \right) \right] dx. \end{aligned}$$

Thus,

$$H_1 - H_2 = \int_{x=-\infty}^{\infty} -\varphi_t \left(x, \frac{x+1}{k} \right) + k\varphi \left(x, \frac{x+1}{k} \right) - k\varphi \left(x, \frac{x}{k} \right) dx. \tag{3.22}$$

Next

$$H_3 = \int_{x=-\infty}^{\infty} \int_{t=\frac{x-1}{k}}^{\frac{x}{k}} kt\ \varphi_{tt}\ dtdx,$$

and by integration by parts, we get

$$\begin{aligned} H_3 &= \int_{x=-\infty}^{\infty} \left[x\varphi_t\left(x, \frac{x}{k}\right) - \left(\frac{x-1}{k}\right)\varphi_t\left(x, \frac{x-1}{k}\right) - \int_{t=\frac{x-1}{k}}^{\frac{x}{k}} k\varphi_t\ dt \right] dx \\ &= \int_{x=-\infty}^{\infty} \left[x\varphi_t\left(x, \frac{x}{k}\right) - \left(\frac{x-1}{k}\right)\varphi_t\left(x, \frac{x-1}{k}\right) - \right. \\ &\qquad \left. k\varphi\left(x, \frac{x}{k}\right) + k\varphi\left(x, \frac{x-1}{k}\right) \right] dx \end{aligned}$$

and

$$\begin{aligned} H_4 &= \int_{x=-\infty}^{\infty} x \int_{t=\frac{x}{k}}^{t=\frac{x-1}{k}} \varphi_{tt}\ dtdx \\ &= \int_{x=-\infty}^{\infty} x \left[\varphi_t\left(x, \frac{x}{k}\right) - \varphi_t\left(x, \frac{x-1}{k}\right) \right] dx\ . \end{aligned}$$

Thus,

$$H_3 - H_4 = \int_{x=-\infty}^{\infty} \varphi_t\left(x, \frac{x-1}{k}\right) - k\varphi\left(x, \frac{x}{k}\right) + k\varphi\left(x, \frac{x-1}{k}\right) dx\ . \quad (3.23)$$

Also, we compute (3.21).

$$\begin{aligned} H_5 &= \int_{x=-\infty}^{\infty} \int_{t=\frac{x-1}{k}}^{\frac{x+1}{k}} \varphi_{tt}\ dtdx \\ &= \int_{x=-\infty}^{\infty} \left[\varphi_t\left(x, \frac{x+1}{k}\right) - \varphi_t\left(x, \frac{x-1}{k}\right) \right] dx, \end{aligned} \quad (3.24)$$

and by summing we obtain

$$J_2 = k \int_{x=-\infty}^{\infty} \left[\varphi\left(x, \frac{x+1}{k}\right) - 2\varphi\left(x, \frac{x}{k}\right) + \varphi\left(x, \frac{x-1}{k}\right) \right] dx. \quad (3.25)$$

Consideration of the support of φ allows us to write Equation (3.25) as

$$J_2 = k\int_{x=-1}^{\infty} \varphi\left(x, \frac{x+1}{k}\right) dx - 2k\int_{x=0}^{\infty} \varphi\left(x, \frac{x}{k}\right) dx + k\int_{x=1}^{\infty} \varphi\left(x, \frac{x-1}{k}\right) dx. \tag{3.26}$$

Using the u-substitutions $u = \frac{x+1}{k}$, $u = \frac{x}{k}$, and $u = \frac{x-1}{k}$ respectively in the three integrals in Equation (3.26) we get

$$J_2 = k^2 \int_{u=0}^{\infty} \varphi(ku-1, u)du - 2k^2 \int_{u=0}^{\infty} \varphi(ku, u)du + k^2 \int_{u=0}^{\infty} \varphi(ku+1, u)du \tag{3.27}$$

Then from (3.18) and (3.27), we can say that

$$J_2 = k^2 J_1,$$

and since we can easily find a φ that makes $J_1 \neq 0$, and since $J_1 = J_2$ is the requirement for U_k to be a solution, we must have

$$k \pm 1.$$

4 Our composite materials problem

We wish to study some of the properties, especially bounds on the gradient, of weak solutions to the following partial differential equation in $\mathbb{R}^2$:

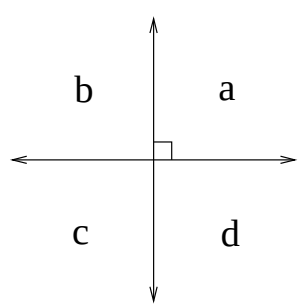
$$\operatorname{div}(A(\vec{x})\nabla U) = 0 \tag{4.1}$$

We assume that $A(\vec{x})$ is a piecewise-constant matrix-valued function of our location in the plane. We assume that for all $\vec{x} \in \mathbb{R}^2$, $A(\vec{x})$ is a symmetric positive definite matrix. Finally, we make some geometric assumptions about the "cuts" or curves of discontinuity of $A(\vec{x})$, but before explaining these assumptions we explain a little bit of the physical background which might lead to Equation (4.1).

We examine first a situation considered by Yanyan Li and Michael Vogelius. (See [LV].) They assumed that they had a fiber-reinforced composite material with identical cross-sections, that their fibers were smooth and were all made of one material with shear modulus equal to a_0, and that the matrix surrounding the fibers was assumed to have a different and fixed shear modulus which they took to be 1 for convenience. The shear modulus of a

material is a positive number which, roughly speaking, measures its strength. They let u represent the out-of-plane elastic displacement, and by modeling the situation with the equations of linear elasticity, they arrive at Equation (4.1), where they have $A(\vec{x}) = a_0\mathrm{I}$ when $\vec{x}$ is in the fiber and $A(\vec{x}) = \mathrm{I}$ when $\vec{x}$ is in the surrounding matrix. In this setting, the most important quantity from an engineering point of view is the stress. The question Li and Vogelius focused on was whether or not the stress would remain pointwise bounded if the fibers approached each other. This stress is bounded if and only if the gradient of U remains bounded, and so their question became the purely mathematical problem of obtaining good bounds on ∇U. They answered this problem by showing that, indeed, with smooth fibers $|\nabla U|$ is less than a constant depending on the boundary data at every point of the interior of the domain, but that this constant is independent of the distance between the fibers.

We became interested in generalizing this work after first author discussed this problem with Yanyan Li when he had a post-doctoral appointment at Rutgers. Specifically, we now assume that we have 4 different materials with a point common to their boundaries, and ask if there is a compatibility condition which can be imposed in the angles they meet and/or on their shear moduli such that the stress remains bounded at this vertex. The simplest example might be a composite in the plane where each quadrant had its own material with shear moduli a, b, c, and d, as in the figure.



In this case one can show that $ac = bd$ will lead to solutions which necessarily have pointwise bounded stress. This paper attempts to describe this result and generalize it. The first order of business, however, is simply to establish compatibility conditions satisfied by the solution of Equation (4.1) along the "cuts" of $A(\vec{x})$. These conditions are analogs to the condition that the function $U_k(x,t)$ is not a weak solution of the wave equation unless $|k| = 1$.

4.1 Definition of weak solution and compatibility conditions

By a weak solution to Equation (4.1) in B_1, we will mean a piecewise differentiable function U such that for any smooth function φ which vanishes outside a compact subset of B_1 we have

$$\int_{B_1} \nabla\varphi(A(\vec{x})\nabla u)d\vec{x} = 0. \tag{4.2}$$

As in the definition of weak solution to the wave equation, this definition does not imply that the divergence of $A(\vec{x})\nabla U$ exists, and so less regularity is assumed for weak solutions of (4.1) than for classical solutions. Once again, however, it is a simple matter to show that wherever $\mathrm{div}(A(\vec{x})\nabla U)$ is continuous, it must equal zero just as in Proposition 3.1. So weak solutions with enough regularity are always classical solutions. Where ∇U has jump discontinuities, it must obey compatibility conditions which are analogs of the "Velocity = 1" condition for our weak solutions of the wave equation. We turn to the derivation of these conditions now.

We want to find compatibility conditions for weak solutions of

$$\mathrm{div}(A(x)\nabla U) = 0 \text{ in } B_1$$

at points where $A(x)$ has jump discontinuities. Before we can begin this pursuit we need to gather some of the more basic properties of the equation, which we do here:

Proposition 4.1. *Suppose for convenience that*

$$A(\vec{x}) = \begin{cases} a & y \geq 0 \\ b & y < 0 \end{cases}$$

Then

(a) Equation (4.1) is Linear

(b) $U(x,y) = c_0 + c_1 x$ is a weak solution of Equation (4.1) for any c_0 and $c_1 \in \mathbb{R}$

(c) If $U(x,y)$ is a weak solution of (4.1) with $U(0,0) = 0$, then $U_\epsilon(x,y) = \epsilon^{-1}U(\epsilon x, \epsilon y)$ is also a weak solution of (4.1) for all $\epsilon, 0 < \epsilon < 1$.

(d) *If U_n is a sequence of weak solutions of Equation (4.1) in B_1 and U_n converges to U uniformly in B_1 and ∇U_n converges to ∇U uniformly in B_1, then U is a weak solution of Equation (4.1)*

Proof.

(a) First item is obvious for classical solutions, and the property of linearity is preserved when considering weak solutions simply by observing the linearity of the integrals involved.

(b) The second item is verified by direct computation. Since $U_x = c_1$ and $U_y = 0$, Equation (4.2) can be written as

$$I := \int_{B_1} (\varphi_x, \varphi_y) \left(A(x) \begin{pmatrix} c_1 \\ 0 \end{pmatrix} \right) d\vec{x}.$$

Then, we compute

$$\begin{aligned} I &= \int_{B_1^+} (\varphi_x, \varphi_y) \cdot (ac_1, 0) d\vec{x} + \int_{B_1^-} (\varphi_x, \varphi_y) \cdot (bc_1, 0) d\vec{x} \\ &= ac_1 \int_{B_1^+} \varphi_x d\vec{x} + bc_1 \int_{B_1^-} \varphi_x d\vec{x} \\ &= ac_1 \iint_{B_1^+} \varphi_x dxdy + bc_1 \iint_{B_1^-} \varphi_x dxdy. \end{aligned}$$

Since φ vanishes on the boundary of B_1, we can conclude after integrating in the x direction,

$$I = ac_1 \int_0^1 (0-0)dy + bc_1 \int_{-1}^0 (0-0)dy = 0.$$

Thus $U(x, y) = c_0 + c_1 x$ is a weak solution of Equation (4.1) for any c_0 and $c_1 \in \mathbb{R}$.

(c) For the third item, we let $\vec{x}' := \epsilon \vec{x}$ so that $x' = \epsilon x$, $y' = \epsilon y$, and $dA(\vec{x}') = \epsilon^2 dA(\vec{x})$. So it follows that $\nabla_x V = \epsilon \nabla_{x'} V$. Now we can check that $U_\epsilon(x, y) = \epsilon^{-1} U(\epsilon x, \epsilon y) = \epsilon^{-1} U(x', y')$ is a weak solution of (4.1)

whenever U is. We assume $\varphi \in C_0^1(B_1)$ is otherwise arbitrary.

$$\begin{aligned} I &= \int_{B_1} \nabla\varphi(\vec{x})A(\vec{x})\nabla U_\epsilon(x,y)dA(\vec{x}) \\ &= \epsilon^{-1}\int_{\vec{x}\in B_1} \nabla_{\vec{x}}\varphi(\vec{x})A(\vec{x})\nabla_{\vec{x}}U(\epsilon x,\epsilon y)dA(\vec{x}) \\ &= \epsilon^{-1}\int_{\vec{x}'\in B_\epsilon} \nabla_{\vec{x}}\varphi(\epsilon^{-1}\vec{x}')A(\epsilon^{-1}\vec{x}')\nabla_{\vec{x}}U(x',y')\epsilon^2 dA(\vec{x}') \\ &= \epsilon\int_{\vec{x}'\in B_\epsilon} \nabla_{\vec{x}}\psi(\vec{x}')A(\vec{x}')\nabla_{\vec{x}}U(x',y')dA(\vec{x}') \end{aligned} \tag{4.3}$$

where in the last equality we used $A(\delta\vec{x}) = A(\vec{x})$ for any $\delta > 0$, and we let $\psi(\vec{x}') := \varphi(\epsilon^{-1}\vec{x}')$, so that our function ψ is an arbitrary function in $C_0^1(B_\epsilon) \subset C_0^1(B_1)$. Continuing with our integration and using the chain rule estimates mentioned above we have

$$I = \epsilon^3 \int_{\vec{x}'\in B_\epsilon} \nabla_{\vec{x}'}\psi(\vec{x}')A(\vec{x}')\nabla_{x'}U(\vec{x}')dA(\vec{x}') \ .$$

Now since ψ vanishes outside B_ϵ, we have

$$I = \epsilon^3 \int_{\vec{x}'\in B_1} \nabla_{\vec{x}'}\psi(\vec{x}')A(\vec{x}')\nabla_{x'}U(\vec{x}')dA(\vec{x}') = 0$$

where the last equality is due to the original assumption that U is a weak solution of Equation (4.1) in B_1. Finally we observe by the original definition of I, this computation confirms that U_ϵ must also be a weak solution in B_1.

(d) For the fourth item, let $\varphi \in C_0^1(B_1)$. Then

$$\begin{aligned} &\int_{B_1} \nabla\varphi(A(x)\nabla U)dx \\ = &\int_{B_1} \nabla\varphi(A(x)(\nabla U - \nabla U_n + \nabla U_n))dx \\ = &\int_{B_1} \nabla\varphi(A(x)(\nabla U - \nabla U_n))dx + \int_{B_1} \nabla\varphi(A(x)\nabla U_n)dx \\ = &\int_{B_1} \nabla\varphi(A(x)(\nabla U - \nabla U_n))dx \end{aligned}$$

Next, we would like to estimate $\left|\int_{B_1} \nabla\varphi(A(x)\nabla U)dx\right|$. By the last computation it is equal to $\left|\int_{B_1} \nabla\varphi(A(x)(\nabla U - \nabla U_n))dx\right|$. Then we can say that

$$\begin{aligned}
&\left|\int_{B_1} \nabla\varphi(A(x)(\nabla U - \nabla U_n))dx\right| \\
\leq\ & \int_{B_1} |\nabla\varphi||A(x)||\nabla U - \nabla U_n|dx \\
\leq\ & \max_{\vec{x}\in B_1} |\nabla\varphi(\vec{x})| \cdot \max\{a,b\} \cdot \max_{\vec{x}\in B_1} |\nabla U(\vec{x}) - \nabla U_n(\vec{x})| \cdot |\pi| \\
\leq\ & C \cdot \max_{\vec{x}\in B_1} |\nabla U(\vec{x}) - \nabla U_n(\vec{x})|
\end{aligned} \tag{4.4}$$

where we note that the constant C is independent of n. Now by combining the last two computations we have

$$\left|\int_{B_1} \nabla\varphi(A(x)\nabla U)dx\right| \leq C \cdot \max_{\vec{x}\in B_1} |\nabla U(\vec{x}) - \nabla U_n(\vec{x})|$$

By taking the limit on both sides as $n \to \infty$ we get

$$\left|\int_{B_1} \nabla\varphi(A(x)\nabla U)dx\right| \leq 0$$

by the uniform convergence of ∇U_n to ∇U. Therefore,

$$\int_{B_1} \nabla\varphi(A(x)\nabla U)dx = 0$$

and so U is weak solution of (4.1). $\square$

At this point, we need one more result related to the smoothness of weak solutions of Equation (4.1). This fact is due to the regularity theory enjoyed by solutions of elliptic partial differential equations and we state it here without proof.

Theorem 4.2. *With $A(\vec{x})$ as above, if $U(x,y)$ is a weak solution of Equation (4.1) in B_1, then U is continuous in B_1, is real analytic on $\overline{B_1^+}$, and is real analytic on $\overline{B_1^-}$.*

(A real analytic function is a function which is equal to its power series, $B_1^+ := B_1 \cap \{y > 0\}$, and $B_1^- := B_1 \cap \{y < 0\}$.)

Armed with Theorem (4.2) and Proposition (4.1) we can quickly produce a necessary compatibility condition satisfied pointwise across the jump discontinuity of $A(\vec{x})$. We let U_y^+ and U_y^- denote the one sided y-derivatives of U along the x-axis.

Proposition 4.3 (Necessary pointwise compatibility condition). *Assume U is a continuous weak solution of Equation (4.1) in B_1 with $A(\vec{x})$ given as in proposition (4.1), then for $-1 < x < 1$ we have*

$$aU_y^+(x,0) = bU_y^-(x,0) \ . \tag{4.5}$$

Proof. By the translation invariance in x along with the rescaling properties of Equation (4.1), it is clear that it suffices to prove this result when $x = 0$. So using Theorem (4.2) we express our solution $U(x,y)$ as follows

$$U(x,y) = \begin{cases} C_0 + C_1 x + C_2^+ y + C_{11}x^2 + C_{12}^+ xy + C_{22}^+ y^2 + \cdots & y \geq 0 \\ C_0 + C_1 x + C_2^- y + C_{11}x^2 + C_{12}^- xy + C_{22}^- y^2 + \cdots & y \leq 0 \end{cases}$$

Obviously, $U_y^+(x,0) = C_2^+$ and $U_y^-(x,0) = C_2^-$, so we need to show $aC_2^+ = bC_2^-$ to prove our theorem.

By using Proposition (4.1)(a) and (4.1)(b), we know $\tilde{U}(x,y) := U(x,y) - C_0 - C_1 x$ is still a weak solution of Equation (4.1). By proposition (4.1)(c), $\tilde{U}_\epsilon(x,y) := \epsilon^{-1}\tilde{U}(\epsilon x, \epsilon y)$ is still a solution. Now a bit of algebra shows

$$\tilde{U}_\epsilon(x,y) = \begin{cases} C_2^+ y + \epsilon C_{11}x^2 + \epsilon C_{12}^+ xy + \epsilon C_{22}^+ y^2 + O(\epsilon^2) & y \geq 0 \\ C_2^- y + \epsilon C_{11}x^2 + \epsilon C_{12}^- xy + \epsilon C_{22}^- y^2 + O(\epsilon^2) & y \leq 0, \end{cases}$$

and $\tilde{U}_\epsilon(x,y) =$ converges to

$$\tilde{U}_0(x,y) = \begin{cases} C_2^+ y & y \geq 0 \\ C_2^- y & y \leq 0 \end{cases}$$

as $\epsilon \to 0$. Since the convergence of $\tilde{U}_\epsilon$ to $\tilde{U}_0$ is uniform in B_1, and since a short estimation shows that $\nabla\tilde{U}_\epsilon$ converges uniformly (and that it converges to $\nabla\tilde{U}_0$), we can now use Proposition (4.1)(d) to conclude that $\tilde{U}_0$ is a weak

solution of Equation (4.1). Knowing that $\tilde{U}_0$ is a weak solution of Equation (4.1) allows us to say that for any $\varphi \in C_0^1(B_1)$, we must have

$$0 = \int_{B_1} \nabla\varphi(A(x)\nabla\tilde{U}_0)dA.$$

On the other hand,

$$\begin{aligned}
&\int_{B_1} \nabla\varphi(A(x)\nabla\tilde{U}_0)dA \\
&= a\int_{B_1^+} (\varphi_x, \varphi_y)\cdot(0, C_2^+)dA + b\int_{B_1^-} (\varphi_x, \varphi_y)\cdot(0, C_2^-)dA \\
&= aC_2^+ \int_{B_1^+} \varphi_y \; dydx + bC_2^- \int_{B_1^-} \varphi_y \; dydx \\
&= aC_2^+ \left(0 - \int_{\{y=0\}} \varphi(x,0) \; dx\right) + bC_2^- \left(\int_{\{y=0\}} \varphi(x,0) \; dx - 0\right) \\
&= \left(-aC_2^+ + bC_2^-\right) \int_{\{y=0\}} \varphi(x,0) \; dx.
\end{aligned}$$

By choosing φ to ensure that the integral does not vanish, we conclude that $aC_2^+ = bC_2^-$. $\square$

So of course we know now that across the x-axis we must satisfy Equation (4.5). We now show that for weak solutions of Equation (4.1), Equation (4.5) is the *only* condition we require on the x-axis.

Proposition 4.4 (Sufficient pointwise compatibility conditions). *Take $A(\vec{x})$ as above. Assume U is a continuous piecewise differentiable function such that $\Delta U = 0$ in B_1^+, $\Delta U = 0$ in B_1^-, and $aU_y^+(x,0) = bU_y^-(x,0)$ for all $x \in (-1,1)$. Then U is a weak solution of Equation (4.1).*

Proof. We let $\varphi \in C_0^1(B_1)$ and use Green's identities and our assumptions

to compute

$$
\begin{aligned}
&\int_{B_1} \nabla\varphi A(x)\nabla U d\vec{x} \\
&= a\int_{B_1^+} \nabla\varphi\cdot\nabla U d\vec{x} + b\int_{B_1^-} \nabla\varphi\cdot\nabla U d\vec{x} \\
&= -a\int_{B_1^+} \varphi\Delta U d\vec{x} + a\int_{\partial B_1^+} \varphi\frac{\partial U}{\partial n} d\mathcal{H}^1 - b\int_{B_1^-} \varphi\Delta U d\vec{x} + b\int_{\partial B_1^-} \varphi\frac{\partial U}{\partial n} d\mathcal{H}^1 \\
&= a\int_{\{y=0\}\cap\partial B_1^+} \varphi\frac{\partial U}{\partial n} dx + b\int_{\{y=0\}\cap\partial B_1^-} \varphi\frac{\partial U}{\partial n} dx \\
&= \int_{\{y=0\}\cap\partial B_1^+} \varphi(-aU_y^+(x,0))dx + \int_{\{y=0\}\cap\partial B_1^-} \varphi(bU_y^-(x,0))dx \\
&= -\int_{\{y=0\}} \varphi(aU_y^+)dx + \int_{\{y=0\}} \varphi(aU_y^+)dx \\
&= 0. \quad \square
\end{aligned}
$$

4.2 Applying the compatibility conditions to our setting

In the previous section we took $A(\vec{x})$ to equal a in the upper half plane and b in the lower half plane simply for convenience. Obviously we would obtain a similar compatibility condition across any linear "cut" of $A(\vec{x})$. If we now take

$$
A(\vec{x}) = \begin{cases} a & 0 \le \theta < \frac{\pi}{2} \\ b & \frac{\pi}{2} \le \theta < \pi \\ c & \pi \le \theta < \frac{3\pi}{2} \\ b & \frac{3\pi}{2} \le \theta < 2\pi \end{cases} \tag{4.6}
$$

(with $A(0,0)$ defined arbitrarily), then except for the behavior at the origin the following theorem is proven with the exact same arguments as Proposition (4.3) and Proposition (4.4).

Theorem 4.5. *Take $A(\vec{x})$ as defined in Equation (4.6), and assume $U(\vec{x})$ is a continuous function which is piecewise differentiable and whose derivatives are bounded. Then $U(\vec{x})$ is a weak solution of Equation (4.1) if and only if*

(a) $\Delta U = 0$ within each open quadrant, and

(b) U satisfies the compatibility conditions

$$\begin{aligned}
&(i)\ aU_x^+(0,y) = bU_x^-(0,y) && y>0\\
&(ii)\ bU_y^+(x,0) = cU_y^-(x,0) && x<0\\
&(iii)\ cU_x^-(0,y) = dU_x^+(0,y) && y<0\\
&(iv)\ dU_y^-(x,0) = aU_y^+(x,0) && x>0.
\end{aligned}$$

Proof. This theorem is obvious except for potential trouble at the origin. In fact, the necessity of the conditions above follows exactly as before. Therefore, to prove the theorem, we need only to show that if U satisfies both conditions a & b, then U must be a weak solution. So we assume $U(x,y)$ satisfies a & b above and we let $\varphi \in C_0^1(B_1)$ and we want to show

$$\int_{B_1} \nabla\varphi(A(\vec{x})\nabla U)d\vec{x} = 0.$$

Let $\psi(\vec{x})$ be a function which satisfies

1) $\psi \in C^1(\mathbb{R}^2)$

2) $0 \le \psi \le 1$

3) $\psi \equiv 1$ on $B_{\frac{1}{2}}$

4) $\psi \equiv 0$ outside $B_{\frac{3}{4}}$

5) $|\nabla\psi| \le 4$.

(It is a simple exercise to construct such a ψ.) We will use this auxiliary function to separate the integral into a integral *away* from the origin which vanishes from our conditions on U exactly as in the previous proposition, and an integral *near* the origin, which we will be able to make arbitrarily small.

Now observe that for any $0 < \epsilon < 1$, we have

$$\begin{aligned}
&\int_{B_1} \nabla\varphi(\vec{x})(A(\vec{x})\nabla U(\vec{x}))d\vec{x} \\
&= \int_{B_1} \nabla\varphi(\vec{x}) \left(1 - \psi\left(\frac{\vec{x}}{\epsilon}\right) + \psi\left(\frac{\vec{x}}{\epsilon}\right)\right) A(\vec{x})\nabla U(\vec{x})d\vec{x} \\
&= \int_{B_1} \nabla\left(\varphi(\vec{x})\left(1 - \psi\left(\frac{\vec{x}}{\epsilon}\right)\right)\right) A(\vec{x})\nabla U(\vec{x})d\vec{x} \\
&\qquad\qquad + \int_{B_1} \nabla\left(\varphi(\vec{x})\psi\left(\frac{\vec{x}}{\epsilon}\right)\right) A(\vec{x})\nabla U(\vec{x})d\vec{x} \\
&= \int_{B_1\setminus B_{\frac{\epsilon}{2}}} \nabla\left(\varphi_\epsilon^\circ(\vec{x})\right. A(\vec{x})\nabla U(\vec{x})d\vec{x} + \int_{B_\epsilon} \nabla\left(\varphi_\epsilon^i(\vec{x})\right. A(\vec{x})\nabla U(\vec{x})d\vec{x}
\end{aligned}$$

where

$$\begin{aligned}
\varphi_\epsilon^\circ(\vec{x}) &:= \varphi(\vec{x})\left(1 - \psi\left(\frac{\vec{x}}{\epsilon}\right)\right) \\
&\textit{and} \\
\varphi_\epsilon^i(\vec{x}) &:= \varphi(\vec{x})\left(\psi\left(\frac{\vec{x}}{\epsilon}\right)\right)
\end{aligned}$$

The integral over $B_1 \setminus B_{\frac{\epsilon}{2}}$ vanishes after splitting it into the component quadrants, using Green's identity, and applying the compatibility conditions on U. This computation is essentially the same as the computation in the proof of Proposition (4.3). So we are left with the equality

$$\left|\int_{B_1} \nabla\varphi(\vec{x})A(\vec{x})\nabla U(\vec{x})d\vec{x}\right| = \left|\int_{B_\epsilon} \nabla\left(\varphi_\epsilon^i\right) A(\vec{x})\nabla U(\vec{x})d\vec{x}\right|$$

for any $\epsilon > 0$. On the other hand we can estimate

$$\begin{aligned}
&\left|\int_{B_\epsilon} \nabla\left(\varphi_\epsilon^i\right) A(\vec{x})\nabla U(\vec{x})d\vec{x}\right| \\
\leq\ & \int_{B_\epsilon} \left|\nabla\left(\varphi_\epsilon^i\right)\right| |A(\vec{x})| \left|\nabla U(\vec{x})\right| d\vec{x} \\
\leq\ & \max\{a,b,c,d\} \max_{x\in B_1} |\nabla U(\vec{x})| \max_{x\in B_\epsilon} \left|\nabla\left(\varphi_\epsilon^i\right)\right| \cdot \pi\epsilon^2 \\
\leq\ & \max\{a,b,c,d\} \max_{x\in B_1} |\nabla U(\vec{x})| \cdot (C\cdot\epsilon^{-1}) \cdot \pi\epsilon^2 \\
=\ & C\epsilon \max_{x\in B_1} |\nabla U(\vec{x})|
\end{aligned}$$

Since we have assumed ∇U is bounded, and since ϵ can be arbitrarily small, we must have

$$\left| \int_{B_1} \nabla\varphi(\vec{x})(A(\vec{x})\nabla U(\vec{x}))d\vec{x} \right| = 0$$

which proves the theorem. $\square$

4.3 Synthesis of previous results

We can now use Theorem (4.5) and what we know about the separated solutions of Laplace's Equation to attempt to construct a family of orthogonal solutions to Equation (4.1). We are particularly interested in confirming three properties of any family we find:

1. Is the family orthogonal, and if yes, what is the correct notion of dot product for this orthogonality? In other words, is there an analogue of Equation (2.2)?

2. Is the family complete? In other words, is there an analogue of Theorem (2.1)?

3. Do all of the solutions in our family have bounded gradients near the origin?

In the case where $ac = bd$, we have the following family of solutions obtained by "glueing" together the $r^n \cos n\theta$ and $r^n \sin n\theta$ functions in a way which satisfies the compatibility conditions. We define the following functions for $j \geq 0$:

$$\Phi_{1,j}(r,\theta) := \begin{cases} c\,[r^{2j+1}\cos((2j+1)\theta)] & -\pi/2 \leq \theta \leq \pi/2 \\ d\,[r^{2j+1}\cos((2j+1)\theta)] & \pi/2 \leq \theta \leq 3\pi/2 \end{cases}$$

$$\Phi_{2,j}(r,\theta) := \begin{cases} c\,[r^{2j+1}\sin((2j+1)\theta)] & 0 \leq \theta \leq \pi \\ b\,[r^{2j+1}\sin((2j+1)\theta)] & \pi \leq \theta \leq 2\pi \end{cases}$$

$$\Phi_{3,j}(r,\theta) := \left[r^{2j}\ \cos(2j\theta) \right]$$

$$\Phi_{4,j}(r,\theta) := \begin{cases} a^{-1}\,[r^{2j}\sin(2j\theta)] & 0 \leq \theta < \pi/2 \\ b^{-1}\,[r^{2j}\sin(2j\theta)] & \pi/2 \leq \theta < \pi \\ c^{-1}\,[r^{2j}\sin(2j\theta)] & \pi \leq \theta < 3\pi/2 \\ d^{-1}\,[r^{2j}\sin(2j\theta)] & 3\pi/2 \leq \theta < 2\pi\ . \end{cases}$$

It follows by Sturm-Liouville theory that the set above is complete, and by inspection it is clear that the gradients are all bounded in B_1. A long but elementary computation shows that the functions $\Phi_{i,j}(1,\theta)$ are orthogonal in $L^2(S^1; A(1,\theta)\ d\theta)$, and it is also simple to show that

$$\mathbf{div}(A(r,\theta)\nabla\Phi_{i,j}) = 0\ .$$

The space $L^2(S^1; A(1,\theta)\ d\theta)$ is exactly like the space $L^2(S^1)$, except that instead of asking that functions satisfy

$$\int_0^{2\pi} f(\theta)^2\ d\theta < \infty\ ,$$

we require

$$\int_0^{2\pi} f(\theta)^2\ A(1,\theta)\ d\theta < \infty\ .$$

Of course since the constants a, b, c, and d are positive constants bounded away from 0 and ∞, it is clear that the functions in $L^2(S^1; A(1,\theta)\ d\theta)$ are the same as the functions in $L^2(S^1)$. On the other hand, in this space, the dot product between two functions, f and g is defined by

$$f \cdot g := \int_0^{2\pi} f(\theta)g(\theta)\ A(1,\theta)\ d\theta\ ,$$

so the "geometry" of these functions is different.

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