MODELING GROUND EFFECT THROUGH COMPLEX FLUID FLOW ANALYSIS

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Abstract. The aim of this paper is to illustrate ground effect, which is when an airfoil gains extra lift when it moves closer to the ground, through complex analysis. Specifically, using complex fluid flow to evaluate the interaction of the flow between the airfoil and the ground to mathematically explain ground effect.

1. Introduction

It is a known phenomenon that when an airplane flies at a certain distance to the ground, the plane experiences an increase in its lift, making it easier to fly. This occurrence is referred to as the ground effect, and it has many explanations, including the compression of air and yet others involving wing tip vortices. However, in this paper the motivation was not to explain ground effect but rather to model it using complex fluid flow.

2. Conformal Mapping

The goal here is to follow the treatment in Fisher's *Complex Variables* p. 246-259, of the Jukowski airfoil, with the additional requirement that the ground, i.e. a horizontal line, is also a streamline.

So let $\mathbb{H} = \{z : \ \operatorname{Im} z > 0\}$, and let D be a disk such that $i \in D \subset \mathbb{H}$. Actually, for $0 < r < 1$, we will let

$$D = D_r = \left\{ z \in \mathbb{H} : \ \left| \frac{z-i}{z+i} \right| \leq r < 1 \right\}$$

The author(s) was(were) partially supported by the Kansas State University REU NSF grant GOMT530725.

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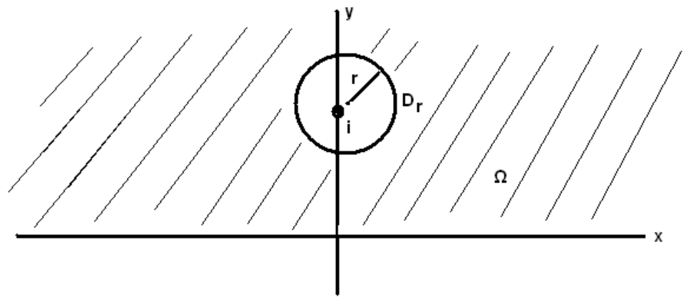
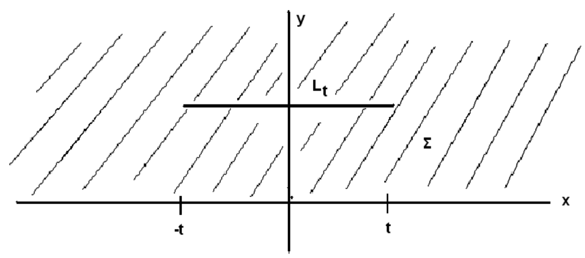


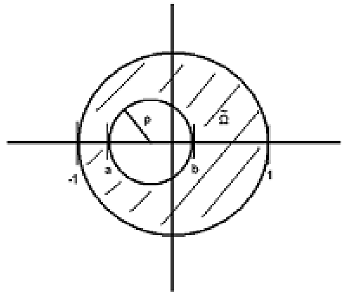
Check that the circles ∂D_r foliate the upper half-plane as r varies between 0 and 1.

We want to find an analytic and one-to-one map F of the domain $\Omega = \mathbb{H} \setminus D_r$ onto the domain $\Sigma = \mathbb{H} \setminus L_t$, where

$$L_t = \{z = x + i \in \mathbb{H} : \ x \in [-t, t]\}$$



We will first compute the harmonic function $\psi = \operatorname{Im} F$. Notice that $\psi = 0$ on $\mathbb{R}$, $= 1$ on ∂D_r, is > 0 on Ω, and tends to ∞ at ∞. We give ψ as the sum of two functions: $\psi = \omega + ch$, for some $c > 0$. The function ω is the unique bounded harmonic function which solves the Dirichlet Problem on Ω with 0 on $\mathbb{R}$ and 1 on ∂D_r. The function h is the unique (up to positive multiple) positive harmonic function on Ω which is 0 on $\mathbb{R}$ and on ∂D_r. In order to write down formulas for these two functions we change variables and apply the linear fractional transformation $(z - i)/(z + i)$ which sends $\mathbb{H}$ onto $\mathbb{D} = \{|z| < 1\}$, and sends Ω to the annulus $\tilde{\Omega} = \{r < |w| < 1\}$. This transformation sends the real line to the unit circle, with ∞ going to 1, but unfortunately it does not send D_r to a concentric circle, so another transformation is required to do this.



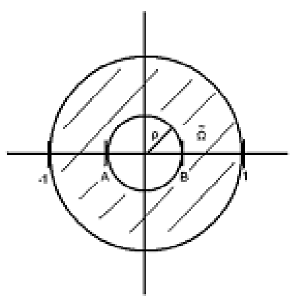
The needed transformation is of the form:

$$W = \frac{z - \lambda}{1 - \lambda z}$$

where λ is:

$$\lambda = \frac{1 + ab - \sqrt{(1-a^2)(1-b^2)}}{a+b}$$

This gives you the annulus needed:



In these new variables the function

$$\tilde{\omega}(w) = \frac{\log |w|}{\log r}$$

is harmonic, is $= 0$ on $\{|w| = 1\}$ and is $= 1$ on $\{|w| = r\}$. So

$$\omega(z) = \tilde{\omega}\left(\frac{z-i}{z+i}\right).$$

Likewise,

$$h(z) = \tilde{h}\left(\frac{z-i}{z+i}\right)$$

where $\tilde{h}$ is positive harmonic on the annulus $\tilde{\Omega}$, is $= 0$ on $\partial\tilde{\Omega} \setminus \{1\}$, but tends to infinity as w tends to 1 from within $\tilde{\Omega}$. Such function is known as the Poisson kernel at 1 for the annulus $\tilde{\Omega}$. A formula is given in Henrici's book Vol. III, p. 229-230.

3. Initial Analysis

The equation for $\tilde{\omega}$ as given by Henrici is:

$$\tilde{\omega}(z) = \frac{\log|z|}{\log\rho}$$

And $h(\tilde{z}) is$:

$$\tilde{h}(z) = \frac{\log|z|}{\log\rho} + \sum_{m\neq 0} \rho^{|m|} \frac{r^m - r^{-m}}{\rho^m - \rho^{-m}} e^{2\pi i m\sigma}$$

The solution for the Poisson kernel at 1 for the disk is:

$$PoissonKernel = \frac{1-|z|^2}{|1-z|^2}$$

For the remainder of this paper ρ is the inner radius of the annulus calculated from the initial airfoil, and σ is the angle of the stagnation points from the origin, both of which are given initially.

Then the equation for $\tilde{\Psi}$ is as follows:

$$\tilde{\Psi} = c\frac{1-|z|^2}{|1-z|^2} + (1-c)\frac{\log|z|}{\log\rho} - c\sum_{m\neq 0} \rho^{|m|} \frac{r^m - r^{-m}}{\rho^m - \rho^{-m}} e^{2\pi i m\sigma}$$

4. Making F Analytic

When trying to construct an analytic function F, the constant c in $\Psi = \omega + ch$ cannot be chosen arbitrarily. It is the unique choice of c which will make the period of ψ around D_r equal to zero. The period of ψ is defined to be

$$\int_{\partial D_r} \frac{\partial\psi}{\partial\eta} ds$$

where $\partial/\partial\eta$ is the outer normal derivative, i.e. the period of ψ is the circulation of the flow around D_r.

However, we can use ρ and σ to evaluate and solve for c. In order to do this, $\tilde{\Psi}$ can be written in terms of r instead of z, where $r = |z|$, and you get:

$$\tilde{\Psi} = c\frac{1-r^2}{1+r^2-2r\cos(2\pi i\sigma)} + (1-c)\frac{\log r}{\log\rho} - c\sum_{m=1}^{\infty}\rho^m\frac{r^m-r^{-m}}{\rho^m-\rho^{-m}}2\cos(2\pi i\sigma m)$$

Now, c can be factored out of $\tilde{\Psi}$, then you can compute $\frac{\partial\tilde{\psi}}{\partial r}$ and when $\tilde{\Psi} = 0$, $r = \rho$, and $\sigma = \sigma_0$,

$$c = \frac{\frac{1}{\rho\log\rho}}{\frac{2\rho(1+\rho^2-2\rho\cos(2\pi i\sigma_0))+(1-\rho^2)(2\rho-2\cos(2\pi i\sigma_0))}{(1+\rho^2-2\rho\cos(2\pi i\sigma_0))^2} + \frac{1}{\rho\log\rho} + \sum_{m=1}^{\infty} m\rho^m\frac{\rho^m+\rho^{-m}}{\rho^m-\rho^{-m}}2\cos(2\pi\sigma m)}.$$

5. Relating $\tilde{\Psi}$ to Ψ

To account for the velocity of the flow which is a given number and known as v, $\tilde{\Psi}$ is multiplied by a constant a, where $a = \frac{v}{c}$. This gives

$$\tilde{\Psi} = v\frac{1-|z|^2}{|1-z|^2} + (a-v)\frac{\log|z|}{\log\rho} - v\sum_{m=1}^{\infty}\rho^m\frac{r^m-r^{-m}}{\rho^m-\rho^{-m}}2\cos(2\pi i\sigma m)$$

But now we notice that $\tilde{\Psi}$'s components can be written as the real parts of certain functions:

$$\frac{1-|z|^2}{|1-z|^2} = Re(\frac{1+z}{1-z})$$

$$\log|z| = Re(\log z)$$

$$\sum_{m=1}^{\infty}\rho^m\frac{r^m-r^{-m}}{\rho^m-\rho^{-m}}2\cos(2\pi i\sigma m) = Re(\sum_{m=1}^{\infty}\rho^m\frac{z^m-z^{-m}}{\rho^m-\rho^{-m}})$$

and $\tilde{\Psi}$ can be written as the sum of the real parts of these functions which gives

$$\tilde{\Psi} = Re(v\frac{1+z}{1-z} + (a-v)\frac{\log z}{\log\rho} - v\sum_{m=1}^{\infty}\rho^m\frac{z^m-z^{-m}}{\rho^m-\rho^{-m}}).$$

And now this function whose real parts equals $\tilde{\Psi}$ can be pulled out and labeled $g(z)$:

$$g(z) = a\frac{(1+z)}{(1-z)} + (a-v)\frac{\log z}{\log\rho} - v\sum_{m=1}^{\infty}\rho^m\frac{z^m-z^{-m}}{\rho^m-\rho^{-m}}$$

However, we are primarily interested in Ψ which is $\tilde{\Psi}(\frac{z-i}{z+i})$, so we need $g(\frac{z-i}{z+i})$.

$$g(\frac{z-i}{z+i}) = a\frac{z}{i} + (a-v)\frac{\log\frac{z-i}{z+i}}{\log\rho} - v\sum_{m=1}^{\infty}\rho^m\frac{(\frac{z-i}{z+i})^m-(\frac{z-i}{z+i})^{-m}}{\rho^m-\rho^{-m}}$$

And thus

$$\Psi = Re(a\frac{z}{i} + (a-v)\frac{\log\frac{z-i}{z+i}}{\log\rho} - v\sum_{m=1}^{\infty}\rho^m\frac{(\frac{z-i}{z+i})^m-(\frac{z-i}{z+i})^{-m}}{\rho^m-\rho^{-m}})$$

6. Relating Ψ to F

Now we have $\Psi = Re(g(\frac{z-i}{z+i}))$, and we can relate this to our complex potential of the flow F, so that we can evaluate F as in Fisher's *Complex Variables* to find the lift.

$$F = \Phi + i\Psi$$

$$iF = i\Phi - \Psi$$

$$-iF = \Psi - i\Phi$$

$$Re(-iF) = \Psi$$

$$\Psi = Re(g(\frac{z-i}{z+i}))$$

$$Re(-iF) = Re(g(\frac{z-i}{z+i}))$$

Thus our stream function F is therefore $ig(\frac{z-i}{z+i})$ plus a constant.

$$F = az + i(a-v)\frac{\log \frac{z-i}{z+i}}{\log \rho} - iv\sum_{m=1}^{\infty} \rho^m \frac{(\frac{z-i}{z+i})^m - (\frac{z-i}{z+i})^{-m}}{\rho^m - \rho^{-m}} + K$$

7. Solving for Lift

According to Fischer, in order to find the lift we find F', then $(F')^2$. From that the lift of the airfoil is defined as $\int F'^2 dz$ and is evaluated by using the residue theorem to find the coefficient on the term $\frac{1}{z-i}$ which gives the lift around i (since i was contained initially in our region D) giving

$$L = \frac{2i(a-v)}{\log \rho} - \frac{i(a-v)^2}{(\log \rho)^2} - \frac{3}{2}v^2 \sum_{m=1}^{\infty} \frac{(m^2+m)\rho^{2m+1}}{\rho^{2m+1} - \rho - \frac{1}{\rho} + \rho^{-2m-1}}.$$

8. Does it Work?

We were able to find equations that would allow us to solve for the lift of an airfoil closer to the ground. However, the initial conformal mappings that were discussed in the beginning that send Ω to the annulus $\tilde{\Omega}$ involve many linear fractional transformations that make it difficult to solve accurately by hand. As of now, to the best of our knowledge, there is not a program available that would do what we need it to. However, that doesn't mean that all our work is in vain. We believe we have all of the equations needed to solve for the lift once the original airfoil is mapped to the annulus. One way that we might be able to check and see if our equations yield results that agree with the ground effect is to pick some simple airfoil that will give us simple values for ρ σ, then use our equations to calculate the lift. Then, move the airfoil up by some arbitrary amount, say t, and try to calculate ρ_t and σ_t, and then run these new values through our equations to get the new lift. Then, if the inequality $Lift \geq Lift_t$ holds it would not prove that our method works, but it certainly would not disprove it either. More work is definitely needed to determine whether these equations yield accurate results. And once done, more evaluation can be done to see if this method is an accurate way to model ground effect.