

Variation on a theme

Problems:

1. Define the following functional on pairs of twice differentiable functions:

$$F(x, y) := \int_a^b L(t, x(t), y(t), \dot{x}(t), \dot{y}(t), \ddot{x}(t), \ddot{y}(t)) dt.$$

Find the equations that sufficiently regular minima of this functional must satisfy.

2. Find the equations of motion of a pendulum hung on a spring.
3. Find the equations of motion of a small block on a movable wedge.
4. Find the equations of motion of a spherical pendulum.
5. Add a spring to the front of the inverted pendulum cart and find the resulting equations of motion. Use this to determine the equations of motion of an inverted pendulum cart with a force u applied to the rear of the cart.
6. Linearize the equations of the forced inverted pendulum cart about $x = 0$, $\theta = 0$ and $u = 0$.
7. Set the mass of the cart base to be 3 Kg, the mass of the bob to be 1 Kg, the length of the pendulum to be 1 M and use $g = 9.8 \text{ Ms}^{-2}$. After taking the Laplace transform the linearized equation may be written in the form $A(\hat{x}, \hat{\theta}) = s(\hat{x}, \hat{\theta})$. Solve for the coefficients of x and θ in the linearization of u so that the eigenvalues of the matrix A are -3 , -4 , -20 and -21 . This procedure is called pole placement.
8. Consider a cable with linear mass density λ of length ℓ attached to two poles of height a and b a distance L apart. Write an expression for the gravitational potential energy of this cable. What unconstrained functional would you have to minimize to find the smallest energy shape?
9. A string with fixed ends can be modeled so that the potential energy is just a constant k times the length of the string minus the rest length of the string. Assuming that the linear density of the string is given by λ and that the string has length L and that the equation of the string is given by $u = u(x, t)$ write out the kinetic energy of the string and the Lagrangian of the string. Simplify your expression by using the Taylor formula to simplify the square roots.

10. Write out the Euler-Lagrange equations for the following functional of a function of 2 variables:

$$F(u) = \int_a^b \int_c^d L(u(x, t), u_x(x, t), u_t(x, t)) dx dt.$$

Apply the resulting equation to obtain the equation of a vibrating string.

11. Recall that the surface area of a surface of revolution is given by $S(y) := 2\pi \int_a^b y(1 + y_x^2)^{\frac{1}{2}} dx$.
- Write out the Euler Lagrange equations for the stationary points of this functional.
 - Some might suggest that you compute $\frac{d}{dt}(y_x L_{y_x} - L)$. Do so in terms of L and use the Euler Lagrange equations with L 's to obtain a first order equation under the assumption that $L_x \equiv 0$.
 - Use the surface area Lagrangian to derive a first order ODE for the shape of the minimal surface of revolution.
 - Solve the ODE.
12. Figure out where the magic came from. Let $S(y) = \int_a^b L(y, y_t) dt$ (note L is NOT a function of t).
- Substitute $t = t(s)$ and $y = y(t(s)) = y(s)$ to obtain the integral in the form $S(y) = \int_{s_0}^{s_1} K(y, y_s, t_s) ds$.
 - Write out the Euler Lagrange equation corresponding to the t -variable.
 - In the case that L is kinetic energy minus potential energy (think about a falling ball) what does $y_t L_{y_t} - L$ represent?