1. (2 points each) Circle either "True" or "False" for each of the following:

a) True False: Arrow's theorem implies that there is no voting system for three or more alternatives (other than a dictatorship) that satisfies both the Condorcet winner criterion and independence of irrelevant alternatives. Mini-Arrow (version of Arrow in our book)

b) True False: Consider an election with an odd number of voters and two candidates. Majority rule is the only voting system that satisfies anonymity (all voters treated the same), neutrality (switching the order of the alternatives switches the result), and monotonicity. May's theorem

c) True False: If a voting system satisfies monotonicity, then a winning alternative will still be a winner if a voter improves the ranking of this winning alternative on his ballot.

d) True False: Consider the weighted voting system [q: w(A), w(B), w(C)] = [5: 3, 2, 1] (A B C). Then B and C are both critical voters in the blocking coalition {B, C}. C not critical {A,C} has 4 votes; not winning

e) True False: Consider the weighted voting system [q: w(A), w(B), w(C)] = [5: 3, 2, 1] (A B C). Then A and B each have veto power. {B,C} 2+1 = 3 votes – not enough to pass; {A,C} 3+1 = 4 votes – not enough to pass

2. (5 points each) Show your work in an easy to follow manner. Suppose that you have just been dealt a hand of five cards. No two cards are alike. Calculate the following:

(a) How many different ways can you arrange these five cards?

$5! = 5 \times 4 \times 3 \times 2 \times 1 = 120$ ways

(b) You want to keep three of the five cards. How many different hands of three cards are possible from your original hand of five cards? The order of the cards doesn't matter here.

$$C^5_3 = \frac{5 \times 4 \times 3}{3!} = \frac{5 \times 4 \times \cancel{3}}{\cancel{3} \times 2 \times 1} = \frac{5 \times 4}{2} = 10 \text{ ways}$$

3. (5 points) Are the following two weighted voting systems equivalent? Explain.

System 1: [q: w(A), w(B), w(C), w(D)] = [5: 3, 1, 1, 1] (A B C D)
System 2: [q: w(A), w(B), w(C), w(D)] = [8: 5, 2, 2, 1] (A B C D)

Winning coalis for

Syst 1	(# votes)	Syst 2	(# votes)
{A, B, C, D}	(6 votes)	{A, B, C, D}	(10 votes)
{A, B, C}	(5 votes)	{A, B, C}	(9 votes)
{A, B, D}	(5 votes)	{A, C, D}	(8 votes)
{A, C, D}	(5 votes)	{A, B, D}	(8 votes)
can't win without A		can't win without A	

Same winning coals, so the two voting systems are equivalent. (compare the two lists of winning coals, need to match-up to renaming the voters – see p453).

4. (5 points each) The following questions ask you to describe an election with four alternatives and four voters that illustrates the Condorcet voting paradox.

(a) Construct a table containing a set of preference lists that illustrates this situation.

rank	# ballots 1	1	1	1
1st	A	B	C	D
2nd	B	C	D	A
3rd	C	D	A	B
4th	D	A	B	C

(b) Write a story problem describing the four voter situation above.

Four friends are trying to choose a lunch place:

Jeff prefers sandwiches (A) to burgers (B), burgers (B) to salad (C), and salad (C) to pizza (D).

Tyler prefers burgers (B) to salad (C), salad (C) to pizza (D), and pizza (D) to sandwiches (A).

Toby prefers salad (C) to pizza (D), pizza (D) to sandwiches (A), and sandwiches (A) to burgers (B).

Becky prefers pizza (D) to sandwiches (A), sandwiches (A) to burgers (B), and burgers (B) to salad (C).

(c) Explain why this situation seems paradoxical.

A majority (3/4) of our friends prefer sandwiches (A) to burger (B).
A majority (3/4) of our friends prefer burgers (B) to salad (C).
A majority (3/4) of our friends prefer salad (C) to pizza (D).
So, we'd expect that a majority of our friends would prefer sandwiches (A) to pizza (D). But 3/4 of our friends actually prefer pizza (D) to sandwiches (A) — contrary to what we expect.

5. (5 points each) Consider the following set of preference lists for a four candidate, 17 voter election:

	Number of voters (17)			
Rank	7	5	4	1
First	A	B	C	B
Second	D	A	B	C
Third	C	D	A	A
Fourth	B	C	D	D

For your convenience, this table will be repeated on the next two pages.

(a) Find the winner(s) under plurality voting. Show your work.

1st place votes only: A 7 ← A wins.
B 5+1 = 6
C 4
D 0

		Number of voters (17)			
	Rank	7	5	4	1
3	First	A	B	C	B
2	Second	D	A	B	C
1	Third	C	D	A	A
0	Fourth	B	C	D	D

(b) Find the winner(s) under Borda count. Show your work.

A: $(7\times3) + (5\times2) + (4\times1) + (1\times1) = 21+10+4+1 = 36$ ← A wins

B: $(7\times0) + (5\times3) + (4\times2) + (1\times3) = 0+15+8+3 = 26$

C: $(7\times1) + (5\times0) + (4\times3) + (1\times2) = 7+0+12+2 = 21$

D: $(7\times2) + (5\times1) + (4\times0) + (1\times0) = 14+5+0+0 = 19$

ballots points

(c) Find the winner(s) under sequential pairwise voting with the agenda B, A, C, D. Show your work.

B	A
5+4+1 = 10	7

B	C
5+1 = 6	7+4 = 11

C	D
4+1 = 5	7+5 = 12

so D wins.

(d) Find the winner(s) under the Hare system. Show your work.

From (a), D has fewest 1st place votes – eliminate D

	7	5	4	1
1st	A	B	C	B
2nd	C	A	B	C
3rd	B	C	A	A

A has 7 first place votes

B 5+1=6

C 4 ← fewest

Eliminate C

	7	5	4	1
1st	A	B	B	B
2nd	B	A	A	A

A has 7 first place votes

B 5+4+1=10

Eliminating A leaves B as the only remaining alternative. B wins.

	Number of voters (17)			
Rank	7	5	4	1
First	A	B	C	B
Second	D	A	B	C
Third	C	D	A	A
Fourth	B	C	D	D

(e) Find all pairs of alternatives X and Y where all of the voters prefer X to Y. Do all of the voting systems listed above satisfy the Pareto condition for this particular election? Explain.

The possible pairs are
- AB — some prefer A to B; others B to A
- AC — some prefer A to C; others C to A
- AD — All prefer A to D
- BC — some prefer B to C; others C to B
- BD — some prefer B to D; others D to B
- CD — some prefer C to D; others D to C.

All voters prefer A to D. Pareto requires that D not be a winner. That's true in (a), (b), (d) above — but not in (c). Here doesn't satisfy Pareto.

(f) Do all of the voting systems listed above satisfy the Condorcet winner criterion for this particular election? Explain.

CWC: If there's a Condorcet winner, it must be the (only) winner of the election.

Do we have a Condorcet winner?

By (c), B beats A, C beats B, and D beats C. } so no Condorcet winner!
By (e), A beats D

(or could check all of the pairs AB, AC, AD, BC, BD, CD)

So nothing to check. All of the voting systems satisfy CWC for this particular election. (Note there are elections where seq pairwise voting fails CWC).

(g) Find the winners under approval voting if the top two candidates are to be elected. Assume that a voter approves of a candidate if that candidate is in first or second place on that voter's preference list.

Suggestion: Write out the resulting ballot table for approval voting before doing your calculation.

	7	5	4	1
A	X	X		
B		X	X	X
C			X	X
D	X			

approval votes:

A	7+5 = 12
B	5+4+1 = 10
C	4+1 = 5
D	7

A, B are the top two vote getters here, so A and B are the winners.

6. Consider the following weighted voting system:

$$[q\text{: } w(A), w(B), w(C), w(D)] = [5\text{: } \overset{A}{3}, \overset{B}{1}, \overset{C}{1}, \overset{D}{1}]$$

(a) (6 points) Calculate the Banzhaf index for this voting system. Show your work.

winning coal	Extra votes	Critical voters
{A, B, C}	0	A, B, C
{A, C, D}	0	A, C, D
{A, B, D}	0	A, B, D
{A, B, C, D}	1	A

coals with A, + 2 of B, C, D

$C^3_2 = 3$

can't win if A not in coalition

Banzhaf index $(\overset{A}{2\times 4}, \overset{B}{2\times 2}, \overset{C}{2\times 2}, \overset{D}{2\times 2}) = (8, 4, 4, 4)$

(b) (6 points) Calculate the Shapley-Shubik index for this voting system. Show your work.

pivotal

_ _ A _ $\quad 3! = 6$

pivotal

_ _ _ A $\quad 3! = 6$

} 12 permutations where A is pivotal

Total # of permutations (4 voters) $\quad 4! = 4\times 3\times 2\times 1 = 24$

Index of A is $\frac{12}{24} = \frac{1}{2}$

B, C, D have same weight, so index of B = index of C = index of D = x

$\frac{1}{2} + 3x = 1$ (sum of indices = 1)

$3x = 1 - \frac{1}{2} = \frac{1}{2}$

$x = \frac{1}{6}$

Shapley-Shubik index $= (\frac{1}{2}, \frac{1}{6}, \frac{1}{6}, \frac{1}{6})$

(c) (3 points) Does A have the same relative power in both models? Show your work.

model	Rel power of A
Banzhaf	$\frac{8}{8+4+4+4} = \frac{8}{20} = \frac{2}{5} = \frac{4}{10}$
Shapley-Shubik	$\frac{1}{2} = \frac{5}{10}$

A does <u>not</u> have the same relative power in both models (more power in Shapley-Shubik).

7. An elementary school has a personnel committee consisting of the principal (P) and four teachers (A, B, C, D). The committee approves a motion if a simple majority of the four teachers supports it. If the four teachers are deadlocked, the principal's vote is the deciding vote.

(a) (5 points) List the minimal winning coalitions for this committee.

Teachers only:

{A, B, C}
{A, B, D}
{A, C, D}
{B, C, D}

(should have $C^4_3 = 4$ of those)

Need principal (teachers deadlocked):

{P, A, B}
{P, A, C}
{P, A, D}
{P, B, C}
{P, B, D}
{P, C, D}

(should have $C^4_2 = \frac{4 \times 3}{2 \times 1} = \frac{12}{2} = 6$)

4 teachers

exactly 2 'for' to have teachers deadlocked

(b) (5 points) Is power shared equally by the members of this committee? Explain.

Notice that at least three members of the committee must support a motion in order for it to be approved. It doesn't matter which three (or more) members. If three (or more) of the teachers support it, it will pass independent of the principal (have at least three 'yes' votes on the committee already). If only two teachers support it, it will have three 'yes' votes and pass if the principal supports it — and if the principal is against it, three of the five committee members are against (also, it doesn't pass). So power equally shared.

<u>Alt</u> If add voter(s) to any of the winning coal listed in (a), then no voter in that coal is critical. All voters crit in each of coals in (a).

A is critical in 6 win coals (same for B, C, D)
P is critical in 6 win coals

} all members of committee have same Banzhaf index