Math 551 Name________________________________

Exam 1

1. Define $T : \mathbb{R}^2 \to \mathbb{R}^3$ by $T(X) = AX$ where

$$A = \begin{bmatrix} 1 & 2 \\ 2 & 2 \\ 3 & 4 \end{bmatrix}.$$

Find a basis for the image of T and a basis for the nullspace of T.

2. Use the test for independence to determine whether the following vectors are independent:

$$X_1 = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 0 \end{bmatrix}, \qquad X_2 = \begin{bmatrix} 1 \\ 0 \\ 1 \\ -1 \end{bmatrix}, \qquad X_3 = \begin{bmatrix} 2 \\ -1 \\ 0 \\ 3 \end{bmatrix}.$$

Suppose that B is a 3×4 matrix with rank 1 and that X_1, X_2, X_3 satisfy $BX = 0$. Do these vectors span the nullspace of B?

3. Consider the following subset of $\mathbb{R}^3$:

$$\mathcal{W} = \left\{ \begin{bmatrix} x+z \\ y+2z \\ 3x+y+5z \end{bmatrix} \middle| \, x, y, z \in \mathbb{R} \right\}.$$

Show that $\mathcal{W}$ is a subspace and determine its dimension. Find two different bases for $\mathcal{W}$.

4. Answer the following questions without using a calculator.

$$A = \begin{bmatrix} 2 & 0 & 1 & 0 & 3 & 13 \\ -17 & 1 & -71 & 0 & 16 & 0 \\ 1.6 & 0 & 13 & 1 & 4 & 0 \end{bmatrix}$$

(a) What is the rank of A? Explain.

(b) What is the dimension of the nullspace of A?

(c) Will the equation $AX = B$ be solvable for all B?

(d) Will the equation $AX = B$ have at most one solution? Explain.

(e) Find two different bases for the column space of A. Use some theorems from linear algebra to justify your answers.

5. Consider the following matrix:

$$A = \begin{bmatrix} 2 & 1 & 3 & 1 \\ 1 & 1 & 3 & 0 \\ 0 & 1 & 2 & 1 \\ 3 & 3 & 8 & 2 \end{bmatrix}.$$

(a) Compute the *reduced* echelon form of A. Call it R.

(b) Use your answer to find a basis for the row space of A.

(c) Express each row of A as a linear combination of these basis elements.

(d) Do the columns of R span the column space of A? Explain.

6. Answer true or false to each of the following questions. Correct answers without explanation will receive only half credit.

   (a) The solution set to a consistent, rank-two system in four unknowns would be a line in four dimensional space.

   (b) The nullspace of a 3×4 matrix cannot consist of only the zero vector.

   (c) Suppose that A is an $n \times n$ matrix whose reduced form is the $n \times n$ identity matrix I. Then A has independent columns.

   (d) The nullspace of a non-zero 4×4 matrix cannot contain four independent vectors.

   (e) Suppose that A is a 4×9 matrix such that the column space of A is a line in $\mathbb{R}^4$. Then every row of A is a multiple of the first row of A, provided that the first row is non-zero.

   (f) A linear transformation of $\mathbb{R}^2$ into $\mathbb{R}^2$ which transforms $[1,2]^t$ to $[7,3]^t$ and $[3,4]^t$ to $[-1,1]^t$ will also transform $[5,8]^t$ to $[13,7]^t$.

   (g) Suppose that A is invertible and B is any matrix such that AB is defined. Then AB and B have the same nullspace.