Math 551 Name_____

Final Exam July 27, 2001

(20) 1. Suppose that A is a 3×3 matrix with eigenvalues 0, 1, and 2. Find each of the following.

(a) The rank of A.

(b) The determinant of $A^t A$.

(c) The determinant of $A + I$.

(d) The eigenvalues of $(A + I)^{-1}$.

(20) 2. Suppose the elementary operation(eliminations) reduce the matrix A and B to the same row echelon form.

$$R = \begin{bmatrix} 1 & 2 & 0 & 7 \\ 0 & 0 & 1 & 5 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

(a) Which of the following subspaces are sure to be the same for both A and B?

i. Column space

ii. Nullspace

iii. Row space

(b) Each time the subspaces in part (a) are the same for A and B, find a basis for that subspace.

(c) True or False (A is any matrix and X, Y are two vectors): if AX and AY are linearly independent then X and Y are linearly independent.

(20) 3. Let

$$B = \begin{bmatrix} 1 & -1 \\ 2 & -1 \end{bmatrix}.$$

Find the diagonal matrix D such that $B = QDQ^{-1}$. Use this factorization to compute B^4.

(20) 4. Consider the following matrix;

$$A = \begin{bmatrix} 2 & 1 & 3 & 1 \\ 1 & 1 & 3 & 0 \\ 0 & 1 & 2 & 1 \\ 3 & 3 & 8 & 2 \end{bmatrix}.$$

(a) Compute the *reduced* row echelon form of A. Call it R.

(b) Use your answer to find a basis for the row space of A.

(c) Express each row of A as a linear combination of these basis elements.

(d) Do the columns of R span the column space of A? Explain.

(20) 5. Each row of the table below summarizes data collected for some matrix A and the corresponding transformation. The "always solvable" column refers to whether the system $AX = B$ is solvable for all B and the "unique solution" column referes to whether or not there will be at most one solution. Your assignment is to fill in all the missing data from the table. Where the data is incosistent, write "impossible."

Size	Always Solvable	Unique Solution	Dimension of Image	Dimension of Nullspace	Rank
4×3	yes				
3×4	yes				
5×5				1	
5×5			5		
3×2		yes			
4×4	yes				
5×4					3
5×4					5

(20) 6. Let

$$A = \begin{bmatrix} 4 & -1 & 3 \\ 0 & 2 & 1 \\ 0 & 0 & 3 \end{bmatrix}.$$

(a) Find the charactersitic polynomial of A.

(b) Find the eigenvalues of the matrix A.

(c) Is A diagonalizable? Explain.

(d) If your answer to part (c) was "yes," then find a matrix Q that diagonalizes A.

(20) 7. Let $\{X, Y\}$ be a basis of a subspace $\mathcal{W}$ of $\mathbb{R}^2$ where

$$X = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \qquad \text{and} \qquad Y = \begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix}.$$

Use the Gram-Schmidt process to transform the basis into

(a) orthogonal basis.

(b) orthonormal basis.

(20) 8. Let

$$A = \begin{bmatrix} 1 & 2 \\ -2 & 5 \end{bmatrix}.$$

Find a lower triangular matrix M and an invertible matrix Q such that $A = QMQ^{-1}$.

(20) 9. Express the equation $3x^2 + 3y^2 - 4xy = 1$ in standard form. Give an orthonormal basis which defines the coordinates.

(20) 10. Answer true or false to each of the following questions. Correct answers without explanation will receive only half credit.

(a) A linear transformation of $\mathbb{R}^2$ into $\mathbb{R}^2$ which transforms $[1, 2]^t$ to $[7, 3]^t$ and $[3, 4]^t$ to $[-1, 1]^t$ will also transform $[5, 8]^t$ to $[13, 7]^t$.

(b) The following matrix is invertible

$$\begin{bmatrix} 1 & 2 & -1 & 4 \\ 2 & 2 & 2 & 2 \\ 1 & 2 & 1 & 0 \\ 4 & 6 & 2 & 6 \end{bmatrix}.$$

(c) Suppose that A is invertible and B is any matrix such that AB is defined. Then AB and B have the same nullspace.

(d) The following polynomial could not be the characteristic polynomial of a symmetric matrix.

$$P(\lambda) = (\lambda^2 - 1)(\lambda^2 + 4)^2$$

(e) The following matrix is orthogonal:

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 1 & 1 \end{bmatrix}.$$

(f) The matrix A is real symmetric, has characteristic polynomial $P(\lambda) = \lambda^3(\lambda - 1)^2(\lambda + 3)$ and the nullspace of A has dimension 2.

(g) The following numbers are equal.

$$\begin{vmatrix} 1 & 2 & 3 \\ 4 & 4 & 4 \\ -1 & 2 & -1 \end{vmatrix} \qquad 4\begin{vmatrix} 1 & 2 & 3 \\ 1 & 1 & 1 \\ -1 & 2 & -1 \end{vmatrix}$$

(h) In the following system, you may assume that the determinant of the coefficient matrix is not zero. Then, $z = 0$.

$$\begin{array}{rcrcrcrcl} 2x & + & 3y & - & 5z & + & 3w & = & 5 \\ 7x & + & y & + & 2z & + & 3w & = & 8 \\ x & + & 4y & + & 4z & + & w & = & 5 \\ 3x & + & 2y & + & 4z & + & 5w & = & 5 \end{array}$$

(i) Suppose that the characteristic polynomial of A is $p(\lambda) = \lambda^3(\lambda-2)(\lambda+3)^2$. Then, the nullspace of A can be at most 2 dimensional.

(j) The following matrix is deficient.

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & -2 & 2 \\ 0 & 0 & 4 \end{bmatrix}$$