Lab time _____ Name _____

Applied Matrix Theory - MATH 551
Final Examination
May 13, 2003

Show all your work in the space provided under each question. Each problem is worth 10 points except for problem 14. which is worth 20 points.

1. Find a linear transformation $T : \mathbb{R}^3 \to \mathbb{R}^3$ which rotates points by an angle of $\frac{\pi}{4}$ radians in a counterclockwise direction around the z-axis when viewed from the positive z-axis.

2. Draw a picture of the indicated set. Label all vertices on the figure you draw.

$$\{su + tv : -1 \leq s \leq 2, \quad -1 \leq t \leq 1\}$$

where $u = (1, 1)$, $v = (0, 2)$.

3. Consider $T : \mathbb{R}^4 \to \mathbb{R}^3$ given by $T(x) = Ax$ where

$$A = \begin{bmatrix} 1 & 3 & 1 & 2 \\ 0 & 1 & 4 & 2 \\ 3 & 10 & 7 & 8 \end{bmatrix}.$$

Find a basis for the nullspace of T and a basis for the image of T.

4. In $M(2,2)$ determine if the matrices $\begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}$, $\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$, $\begin{bmatrix} 1 & 1 \\ 3 & 3 \end{bmatrix}$, and $\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$ are linearly independent.

5. Find a matrix P so that $P^{-1}AP$ is diagonal where $A = \begin{bmatrix} 1 & 3 \\ 3 & 1 \end{bmatrix}$.

6. Let S be a set of vectors contained in $\mathbb{R}^n$. Define

$$S^{\perp} = \{v \in \mathbb{R}^n : v \cdot s = 0 \text{ for every } \ s \in S.\}$$

Prove that $S^{\perp}$ is a subspace of $\mathbb{R}^n$.

7. Find $\det \begin{bmatrix} 1 & 2 & 0 & 3 \\ 0 & 1 & 2 & 3 \\ 1 & -1 & 0 & 2 \\ 1 & 1 & 1 & 1 \end{bmatrix}$.

8. Let $f(x) = x$ on $[-1, 1]$. Find the best approximation to f by functions of the form $c_1 \sin \pi x + c_2 \sin 2\pi x$, that is, find c_1 and c_2 so that $\|f(x) - (c_1 \sin \pi x + c_2 \sin 2\pi x)\|$ is a minimum.

9. Use the Gram-Schmidt process to find an orthogonal basis for the subspace W of $\mathbb{R}^5$ spanned by the vectors

$$\begin{bmatrix}1\\1\\0\\0\\0\end{bmatrix}, \begin{bmatrix}1\\1\\1\\0\\0\end{bmatrix} \text{ and } \begin{bmatrix}0\\0\\1\\1\\0\end{bmatrix}.$$

Although it may be possible to guess the answer, you will receive credit only if you use the Gram-Schmidt procedure.

10. Find the eigenvalues as well as a basis for each eigenspace of the matrix A.

$$A = \begin{bmatrix} -3 & 0 & 2 \\ 2 & -3 & 2 \\ 0 & 0 & -3 \end{bmatrix}$$

11. Use Cramer's rule to find x_3. Although it is possible to do this with other methods, you will receive credit only if you use Cramer's rule. Naturally, you may use other methods to check your work if you desire.

$$\begin{aligned} 2x_1 + x_2 - x_3 &= 3 \\ x_1 + x_2 &= 2 \\ x_1 - x_2 + x_3 &= 0 \end{aligned}$$

12. Let

$$A = \begin{bmatrix} 1 & 1 & 2 & 1 & 1 \\ 0 & 1 & 3 & 1 & 0 \\ 1 & 2 & 5 & 2 & 1 \\ 2 & 3 & 7 & 3 & 2 \end{bmatrix}.$$

Find the rank of A.

13. Two Wichita aerospace industries, Spacely Sprockets and Cogswell Cogs, compete for customers. Each year 40% of the customers of Spacely Sprockets change to Cogswell Cogs and 30% of the customers of Cogswell Cogs change to Spacely Sprockets. Find the transition matrix for this system. What is the equilibrium distribution, that is, in the long run, what percentage of the market will each company have?

14. True or false. No justification required.

_____ 1. Every square matrix is invertible.

_____ 2. If a system of equations has more variables than equations it is always solvable.

_____ 3. If a system has more equations than variables it is never solvable.

_____ 4. The nullspace of an $m \times n$ matrix A is a subspace of $\mathbb{R}^n$.

_____ 5. The dimension of a vector space is the number of elements in a basis.

_____ 6. In an n-dimensional vector space a linearly independent set containing n vectors spans the space.

_____ 7. If A is an $n \times n$ orthogonal matrix and v and w are vectors in $\mathbb{R}^n$ then the angle between v and w is equal to the angle between Av and Aw.

_____ 8. If A and B are $n \times n$ matrices, and if $A \neq 0$, $B \neq 0$ then $AB \neq 0$.

_____ 9. If V is a vector space and $W \subseteq V$ is a subspace, then $0 \in W$.

_____ 10. If A is an $m \times n$ matrix then $n =$ dim columnspace $A +$ dim nullspace A.