Applied Matrix Theory

Math 551 Name _____

Final Exam – May 7, 2001

Show all your work in the space provided under each question. Each problem is worth 10 points.

1. Solve the system of equations.

$$\begin{aligned} x + 2y - 3z &= +5 \\ 3x - y + z &= 1 \\ 5x + 3y - 5x &= 11 \end{aligned}$$

2. Given $A = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$, find A^{-1}.

3. Consider the polynomials $p_1(x) = 1, p_2(x) = x + 1$ and $p_3(x) = (x + 1)^2$ which form a basis for P_2. What are the coordinates of $q(x) = x^2 + 1$ with respect to this basis?

4. Find the LU decomposition of the matrix $A = \begin{bmatrix} 2 & 1 & 3 \\ 2 & 5 & 7 \\ 4 & 2 & 7 \end{bmatrix}$.

5. Define $T : \mathbb{R}^3 \to \mathbb{R}^3$ by $T(v) = Av$ where $A = \begin{bmatrix} 1 & 1 & 2 \\ 0 & 2 & 2 \\ 1 & 3 & 4 \end{bmatrix}$.

   Find a basis for the image of T and a basis for the nullspace of T.

6. Use the Gram-Schmidt process to find an orthogonal basis for the subspace of $\mathbb{R}^4$ spanned by

$$\begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix} \text{ and } \begin{bmatrix} 0 \\ 1 \\ 1 \\ 2 \end{bmatrix}$$

7. On $[-1, 1]$ define $f(x) = \begin{cases} 1 \text{ if } x \geq 0 \\ 0 \text{ if } x < 0 \ . \end{cases}$ Find the best approximation to f by functions of the form $c_1 \sin \pi x + c_2 \sin 2\pi x$; that is, find c_1 and c_2 which minimize $\| f - (c_1 \sin \pi x + c_2 \sin 2\pi x) \|$.

8. Find a matrix P so that $P^{-1}AP$ is diagonal where $A = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$.

9. Compute $\det A$ where $A = \begin{bmatrix} 1 & 2 & 3 & 1 \\ 0 & 1 & 0 & 1 \\ 2 & 0 & 1 & 1 \\ 0 & 2 & 1 & 1 \end{bmatrix}$.

10. Suppose A is an $m \times n$ matrix with the property that the equation $Ax = b$ has at most one solution for each $m \times 1$ matrix b. What is the dimension of the nullspace of A? What is the dimension of the column space of A? What can you say about the relative sizes of m and n? Explain your answers.

11. (a) Find a transformation $T : \mathbb{R}^3 \to \mathbb{R}^3$ which rotates $\mathbb{R}^3$ by $\pi/4$ radius around the z-axis.

(b) Let A be the matrix associated to T as in (a). Is A orthogonal? Give reasons to support your claim.

12. In far, far western Kansas there are two cities: Townsville and Metropolis. Each year 20% of the inhabitants of Townsville move to Metropolis and 30% of the inhabitants of Metropolis move to Townsville. Find the transition matrix for this process. If initially there are 10,000 inhabitants in each town, what is the equilibrium distribution of residents on each planet?

13. Define a transformation $t : \mathbb{R}^2 \to \mathbb{R}^2$ by the rule: $T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right)$ is the result of reflecting $\begin{bmatrix} x \\ y \end{bmatrix}$ in the x-axis and then multiplying by the matrix $A = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}$. Find a matrix which represents T.

14. Find all the eigenvalues and a basis for each eigenspace for the matrix

$$A = \begin{bmatrix} 3 & -1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

15. Let $Q = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 1 \end{bmatrix}$ in $\mathbb{R}^4$. Find a resolution of $v = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \end{bmatrix}$ into a component parallel to Q and a component perpendicular to Q.