From maginnis@math.ksu.edu Fri Dec 19 15:44:34 2003 Date: Fri, 19 Dec 2003
From: John Maginnis ¡maginnis@math.ksu.edu¿ To: sheree ¡math@math.ksu.edu¿ Sub
Exam

Dear Sheree, Here is a copy of my final exam that I just gave in Math 551. John

---

Name _____ Math 551, Applied Matrix Theory

Signature _____ Final Exam, December 19, 2003

Show all work for full credit. The exam is worth 150 points.

(6 pts) 1. Show that the matrices $\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$, $\begin{pmatrix} 2 & 1 \\ 0 & 1 \end{pmatrix}$, and $\begin{pmatrix} 1 & 2 \\ 1 & 1 \end{pmatrix}$ are linearly independent.

(4 pts) 2. Do the matrices $\{\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 2 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 2 \\ 1 & 1 \end{pmatrix}\}$ form a basis for the vector space of 2×2 matrices? Explain.

(15 pts) 3. Solve the system of equations $\{x + 3y = 2, \quad 2x + 5y = 5\}$ using three methods.

(a) Row reduce the augmented matrix.

(b) Use Cramer's Rule (show all work).

(c) Find the inverse of the matrix $A = \begin{pmatrix} 1 & 3 \\ 2 & 5 \end{pmatrix}$, and use this to solve the system.

(10 pts) 4. Compute the determinant of the matrix A using two methods.

$$A = \begin{pmatrix} 2 & 3 & 4 \\ 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix}.$$

(a) Use expansion on the first row.

(b) Use row reduction (you may stop at a triangular matrix).

(5 pts) 5. Using the scalar product $(f(x), g(x)) = \int_{-1}^{1} f(x)g(x)dx$, find the length of the function $y = x^3$.

(6 pts) 6. Let A be the matrix $A = \begin{pmatrix} 1 & 1 & 3 \\ -1 & 1 & -2 \\ 1 & 5 & 7 \end{pmatrix}$.

Use row reduction to find the $A = LU$ factorization, with L a lower triangular matrix and U an upper triangular matrix.

(6 pts) 7. Compute the eigenvalues of the matrix $A = \begin{pmatrix} 1 & 4 \\ -1 & 1 \end{pmatrix}$.

(6 pts) 8. Let $T : R^2 \to R^2$ be a linear transformation of the plane. Assume that $T(1,2) = (3,4)$ and $T(2,3) = (5,6)$. Compute $T(2,5)$.

(5 pts) 9. Show that the plane described by the equation $2x - 3y + 4z = 0$ is a vector subspace of R^3.

(5 pts) 10. Show that $T(f(x)) = xf'(x)$ is a linear transformation of the vector space of polynomials of degree less than or equal to two.

(15 pts) 11. Let A be the matrix $A = \begin{pmatrix} 1 & 3 & 1 & 1 \\ 1 & 3 & 2 & 3 \\ 2 & 6 & 1 & 0 \end{pmatrix}$.

(a) Compute the reduced row echelon form of the matrix A.

(b) Give a basis for the nullspace of A.

(c) Give a basis for the column space of A.

(9 pts) 12. Compute the projection of the vector $\vec{v}_1 = (2, 1, 6)$ onto the plane spanned by the two vectors $\vec{v}_2 = (1, 1, 1)$ and $\vec{v}_3 = (1, 2, 3)$.

(10 pts) 13. Let A be the matrix $A = \begin{pmatrix} 2/3 & 1/3 & 2/3 \\ -2/3 & 2/3 & 1/3 \\ 1/3 & 2/3 & -2/3 \end{pmatrix}$.

(a) Show that the columns of A form an orthonormal basis for R^3.

(b) Compute the inverse matrix A^{-1}. (Hint: Use the fact that the columns are orthonormal.)

(9 pts) 14. Use the method of least squares to find the line $y = mx + b$ which is the best approximate linear model for the three data points $(x, y) \in \{(-1, 9), (0, 10), (1, 12)\}$.

(10 pts) 15. For these two word problems, write a system of linear equations, but DO NOT SOLVE.

(a) A small country produces two goods, oil and steel. Producing one barrel of oil requires using .01 barrels of oil and .03 pounds of steel. Producing one pound of steel requires using .04 barrels of oil and .07 pounds of steel. The country wants to export 10 million barrels of oil and 8 million pounds of steel every year.

(b) Four one-way streets intersect at Courthouse Square. On North Street, 10 cars per minute approach the square and 15 cars per minute leave the square. On South Street, 12 cars per minute approach and 8 cars per minute leave. On East Street, 9 cars per minute approach and 13 cars per minute leave. On West Street, 11 cars per minute approach and 6 cars per minute leave the square.

16. The vegetation of a small country consists of forests and grasslands. If a certain region is now a forest, then there is a 10 % chance that in 300 years the region will change to grassland. If a region is now grassland, there is a 20 % chance that in 300 years the region will change to forest.

(3 pts)(a) Give the transition matrix (also called the probability matrix).

(4 pts)(b) If half of the country is now grasslands, find the fraction of the country that will be grassland in 300 years.

(4 pts)(c) If a certain region is now grassland, find the probability that the region will be forest in 600 years.

(6 pts)(d) In the long term (equilibrium state), what fraction of the country will be grasslands?

(6 pts) 17. Let A and B be two square 3×3 matrices with entries in the real numbers. Assume that $BA = -AB$. Prove that either A or B (or both) is not invertible.

(6 pts) 18. Let A be a square $n \times n$ matrix. Prove that A is invertible if and only if $\lambda = 0$ is not an eigenvalue of A.