Name _____

Applied Matrix Theory - MATH 551
Final Exam
December 17, 2002

Show all your work in the space provided under each question. Each problem is worth 10 points except for problem 14 which is worth 20 points.

1. Find the coordinates of the vector $\begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix}$ with respect to the basis

$\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ of $\mathbb{R}^3$.

2. Given $A = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}$, find A^{-1}.

3. Find a basis for the nullspace of the matrix $A = \begin{bmatrix} 1 & 3 & 2 \\ 1 & 3 & 0 \end{bmatrix}$.

4. Use the Gram-Schmidt process to find an orthogonal basis for the subspace of $\mathbb{R}^4$ spanned by $\begin{bmatrix}1\\0\\0\\0\end{bmatrix}$ $\begin{bmatrix}1\\0\\1\\0\end{bmatrix}$ and $\begin{bmatrix}1\\0\\1\\1\end{bmatrix}$. Although it is easy to guess the answer, you will receive credit only if you show the steps of the Gram-Schmidt process.

5. Compute $\det A$ where $A = \begin{bmatrix} 1 & 3 & 1 & 2 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}$.

6. Define $T : \mathbb{R}^3 \to \mathbb{R}^4$ by $Tv = Av$ where $A = \begin{bmatrix} 1 & 1 & 2 \\ 0 & 2 & 2 \\ 1 & 1 & 2 \\ 1 & 3 & 4 \end{bmatrix}$.

Find a basis for the image of T and a basis for the nullspace of T.

7. On $[-1, 1]$ define $f(x) = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x \leq 0 \end{cases}$.

Find the best approximation to f by functions of the form $c_0 + c_1 \cos \pi x$; that is, find c_0 and c_1 which minimize $\|f - (c_0 + c_1 \cos \pi x\|$.

8. Find a matrix P so that $P^{-1}AP$ is diagonal where $A = \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix}$.

9. Find the LU decomposition of the matrix $A = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

10. Find all eigenvalues and a basis for each eigenspace for the matrix $A = \begin{bmatrix} 1 & -1 & 0 \\ 1 & 3 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

11. In far western Kansas, in Cartoon County, there are two cities: Springfield and Townsville. Each year 10% of the inhabitants of Springfield move to Townsville and 20% of the citizens of Townsville move to Springfield. (There is a road between the cities, but no roads connecting them to the outside world.) Find the transition matrix of this process. In the most recent census, there were $1,000$ people in each town. If there are no births or deaths, what is the equilibrium distribution of residents in each town?

12.(a) Find a transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ which reflects points in the x-axis.

(b) Let A be the matrix corresponding to T, that is, $Tv = Av$ for all $v \in \mathbb{R}^2$. Find A and show that A is orthogonal.

13. Let $v = (1, 1)$, $w = (-1, 1)$ in $\mathbb{R}^2$. Draw a picture of the set

$$\{tu + sv \mid 0 \leq t \leq 1, 0 \leq s \leq 1\}.$$

Label all vertices and intercepts on your picture.

14. True or False.

(a) _____ If $T : \mathbb{R}^3 \to \mathbb{R}^2$ is linear, then $T(0) = 0$.

(b) _____ If $T : \mathbb{R}^2 \to \mathbb{R}^2$ is linear, then the nullspace of T is a subspace of $\mathbb{R}^3$.

(c) _____ If $T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$ then $|Tv| = |v|$ for all vectors $v \in \mathbb{R}^2$.

(d) _____ If $A \in M(2,2)$ then A has an inverse matrix A^{-1}.

(e) _____ The column space of a matrix is the same as the nullspace of the matrix.

(f) _____ If $\mathcal{S}$ and $\mathcal{T}$ are subspaces of a vector space then so is $\mathcal{S} \cap \mathcal{T}$.

(g) _____ Any system $Ax = 0$ is consistent.

(h) _____ If $Ax = b$ has more variables than equations then it is consistent.

(i) _____ If $T : \mathbb{R}^n \to \mathbb{R}^m$ is given by $Tx = Ax$ where A is $m \times n$, then the image of T is the column space of A.

(j) _____ If $A, B \in M(2,2)$ and $A \neq 0$, $B \neq 0$ then $AB \neq 0$.