Applied Matrix Theory - MATH 551 NAME _____

Final Exam

December 11, 2000

Show all your work in the space provided under each question.

1. Find a basis for the nullspace of the matrix $A = \begin{bmatrix} 1 & 2 & 0 \\ 1 & 2 & 1 \end{bmatrix}$.

2. Consider the polynomials $p_1(x) = 2$, $p_2(x) = x^2 + 1$, $p_3(x) = x^2 + 2x + 1$ in $\mathcal{P}_2 =$ the set of all polynomials of degree less than or equal to 2. Are p_1, p_2, p_3 linearly independent? Show your computations.

3. Let $A = \begin{bmatrix} 1 & 0 & 2 \\ 2 & -1 & 3 \\ 4 & 1 & 8 \end{bmatrix}$. Compute A^{-1}.

4. Find the coordinates of the vector $\begin{bmatrix} 0 \\ 2 \\ 0 \end{bmatrix}$ with respect to the basis $\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ of $\mathbb{R}^3$.

5. Define $T : \mathbb{R}^3 \to \mathbb{R}^3$ by $T(v) = Av$ where $A = \begin{bmatrix} 1 & 1 & 2 \\ 0 & 2 & 2 \\ 1 & 1 & 2 \end{bmatrix}$. Find a basis for the image of T and a basis for the nullspace of T.

6. Use the Gram-Schmidt process to find an orthogonal basis for the subspace of $\mathbb{R}^4$ spanned by $\begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}$ $\begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \end{bmatrix}$.

7. Compute $\det A$ where $\quad A = \begin{bmatrix} 2 & 3 & 1 & 4 \\ 0 & 1 & 0 & 1 \\ 0 & 2 & 0 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}$

8. On $[-1, 1]$ define f by $f(x) = 1$ if $x \geq 0$, $f(x) = -1$ if $x < 0$. Compute the best approximation to $f(x)$ by functions of the form $c_1 \sin \pi x + c_x \sin 2\pi x$; that is, find c_1, c_2, which minimizes $\|f(x) - (c_1 \sin \pi x + c_2 \sin 2\pi x)\|$.

9. Let W be the subspace of $\mathbb{R}^4$ spanned by $\begin{bmatrix}1\\0\\1\\0\end{bmatrix}$ and $\begin{bmatrix}2\\1\\2\\1\end{bmatrix}$ Find $\text{proj}_W b$ where $b = \begin{bmatrix}1\\1\\1\\0\end{bmatrix}$.

10. Suppose A is an $n \times n$ matrix and λ is a fixed eigenvalue of A. Let

$$W = \{v \in \mathbb{R}^n : Av = \lambda v\}.$$

Prove that W is a subspace of $\mathbb{R}^n$.

11. Find all the eigenvalues and a basis of each eigenspace for the matrix

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & -1 \\ 0 & 1 & 1 \end{bmatrix}.$$

12. Find a matrix P so that $P^{-1}AP$ is diagonal where $A = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$.

13. In a galaxy far, far away are two planets, $Gykx$ and $Mykx$. Each year 40% of the residents of $Gykx$ move to $Mykx$ and 10% of the residents of $Mykx$ move to $Gykx$; the remainder do not move. If this year there are exactly 1 million residents on each planet, what is the equilibrium distribution of residents on each planet?

14. Suppose $T : \mathbb{R}^n \to \mathbb{R}^m$ is a linear transformation with the property that the equation $T(x) = b$ has a unique solution for every $b \in \mathbb{R}^m$. What is the rank of T? What can you say about the relative sizes of m and n?

15.a) Find a transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ which rotates the plane by $\frac{\pi}{6}$ radians around the origin.

b) Let A be the matrix associated to T as in (a). Is A orthogonal? Give reasons to support your claim.