Math 240: Elementary Differential Equations

Exam 1: Summer 2009

Name: Solutions

Instructor: ____________________

This is a closed book exam. No books, notes or calculators are allowed. You must show all your work to receive credit.

(10) 1. Find all solutions of the differential equation (implicit form is accepted)

$$1 + 2\sqrt{y}e^x + (\frac{e^x}{\sqrt{y}} - 1)\frac{dy}{dx} = 0$$

$$\left.\begin{aligned} \frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}(1 + 2\sqrt{y}\,e^x) = \frac{e^x}{\sqrt{y}} \\ \frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}\left(\frac{e^x}{\sqrt{y}} - 1\right) = \frac{e^x}{\sqrt{y}} \end{aligned}\right\} \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}, \text{ exact.}$$

$$F(x,y) = \int M\,dx = \int (1 + 2\sqrt{y}e^x)\,dx = x + 2\sqrt{y}e^x + C_1(y)$$

$$F(x,y) = \int N\,dy = \int \left(\frac{e^x}{\sqrt{y}} - 1\right)dy = 2\sqrt{y}e^x - y + C_2(x)$$

$$x + 2\sqrt{y}e^x + C_1(y) = 2\sqrt{y}e^x - y + C_2(x)$$

$$C_1(y) = -y, \quad C_2(x) = x$$

$$F(x,y) = x - y + 2\sqrt{y}e^x$$

$$\boxed{x - y + 2\sqrt{y}e^x = C}$$

(10) **2.** Find all solutions to the equation

$$\frac{dy}{dx} + 5y = y^2$$

(A) Bernoulli with $n=2$

Substitution $y = v^{\frac{1}{1-2}} = v^{-1} = \frac{1}{v}$

$$\frac{dy}{dx} = -\frac{1}{v^2}\frac{dv}{dx}$$

$$-\frac{1}{v^2}\frac{dv}{dx} + 5\frac{1}{v} = \frac{1}{v^2} \quad // \text{ Multiply by } -v^2$$

$$\frac{dv}{dx} - 5v = -1 \quad // \text{ This is linear}$$

Integrating factor $\mu(x) = e^{\int -5\,dx} = e^{-5x}$

$$e^{-5x}\frac{dv}{dx} - 5e^{-5x}v = -e^{-5x}$$

$$\frac{d}{dx}(e^{-5x}v) = -e^{-5x}$$

$$e^{-5x}v = -\int e^{-5x}dx = \frac{1}{5}e^{-5x} + C$$

$$v = e^{5x}\left(\frac{1}{5}e^{-5x} + C\right) = \frac{1}{5} + Ce^{5x}$$

$$y = \frac{1}{v} = \left(\frac{1}{5} + Ce^{5x}\right)^{-1}$$

Singular solutions: $y=0$ $\quad 0+5\cdot 0=0$ Checks.

$$\boxed{y = \left(\frac{1}{5}+Ce^{5x}\right)^{-1}, \; y=0}$$

(10) **2.** Find all solutions to the equation

$$\frac{dy}{dx} + 5y = y^2$$

(B) Alternative way – It is also separable!

$$\frac{dy}{dx} = y^2 - 5y$$

$$\frac{dy}{dx} = y(y-5)$$

$$\int \frac{dy}{y(y-5)} = \int dx$$

$$\frac{1}{y(y-5)} = \frac{A}{y} + \frac{B}{y-5} = \frac{Ay - 5A + By}{y(y-5)}$$

$$\left.\begin{array}{l} A+B=0 \\ -5A=1 \end{array}\right\} \Rightarrow \begin{array}{l} A=-1/5 \\ B=1/5 \end{array}$$

$$\frac{1}{5}\int \frac{dy}{y-5} - \frac{1}{5}\int \frac{dy}{y} = \int dx$$

$$\frac{1}{5}\ln|y-5| - \frac{1}{5}\ln|y| = x + C$$

$$\frac{1}{5}\ln\left|\frac{y-5}{y}\right| = x + C$$

$$\frac{y-5}{y} = Ke^{5x}, \quad K \neq 0$$

$y=5$ singular $\Rightarrow$ included if $K \in \mathbb{R}$, $y=0$ singular

$$y - 5 = yKe^{5x}$$

$$y - yKe^{5x} = 5$$

$$\boxed{y = \frac{5}{1-Ke^{5x}}, \quad y = 0}$$

(10) **3.** Find all solutions to the equation

$$\frac{dy}{dx} = \frac{y}{1+x^2}$$

This is separable.

$$\int \frac{dy}{y} = \int \frac{dx}{1+x^2}$$

$$\ln|y| = \arctan x + C$$

$$|y| = e^{C} \cdot e^{\arctan x} = K e^{\arctan x}, \quad K>0$$

$$y = K e^{\arctan x} \qquad K \neq 0$$

$y=0$ singular (checks), can be included if $K \in \mathbb{R}$

$$\boxed{y = K e^{\arctan x}, \quad K \in \mathbb{R}}$$

(15) **4.** Solve the initial value problem

$$\frac{dy}{dx} = \frac{y^2 + 4x^2}{4xy}, \qquad y(1) = 2$$

$$\frac{dy}{dx} = \frac{y^2/x^2 + 4}{4y/x}$$ This is <u>homogeneous</u>

Substitution $y = vx$, $\frac{dy}{dx} = x \cdot \frac{dv}{dx} + v$

$$x\frac{dv}{dx} + v = \frac{v^2+4}{4v}$$

$$x\frac{dv}{dx} = \frac{v^2+4}{4v} - v = \frac{4-3v^2}{4v} = \frac{(2-\sqrt{3}v)(2+\sqrt{3}v)}{4v}$$

$$\int \frac{4v\,dv}{4-3v^2} = \int \frac{dx}{x} \qquad \begin{array}{l} u = 4-3v^2 \\ du = -6v\,dv \end{array}$$

$$\| $$

$$-\frac{2}{3}\int \frac{du}{u} = -\frac{2}{3}\ln|4-3v^2| = \ln|x| + C$$

$$-\tfrac{2}{3}\ln\left|4 - \frac{3y^2}{x^2}\right| = \ln|x| + C$$ plug $x=1, y=2$

$$-\tfrac{2}{3}\ln 8 = 0 + C \qquad C = -\tfrac{2}{3}\ln 8 = -2\ln 2$$

$$-\frac{2}{3}\ln\left|4-\frac{3y^2}{x^2}\right| = \ln|x| - 2\ln 2$$

$$-2\ln\left|4-\frac{3y^2}{x^2}\right| = 3\ln|x| - 6\ln 2$$

$$\ln\left|\left(4-\frac{3y^2}{x^2}\right)^2 x^3\right| = \ln 2^6$$

since x positive

$$\frac{(4x^2-3y^2)^2}{x} = \pm 2^6$$

$$4x^2 - 3y^2 = \pm 2^3\sqrt{x}$$

since $x=1, y=2$

y=2 positive

$$y = \pm\sqrt{\frac{4x^2+8\sqrt{x}}{3}}$$

$$y = \sqrt{\frac{4x^2+8\sqrt{x}}{3}}$$

(15) 4. Solve the initial value problem

$$\frac{dy}{dx} = \frac{y^2 + 4x^2}{4xy}, \qquad y(1) = 2$$

Alternative way: $\frac{dy}{dx} = \frac{y}{4x} + \frac{x}{y}$

$\frac{dy}{dx} - \frac{1}{4x} y = \frac{x}{y}$ Bernoulli with $n=-1$

$y = v^{\frac{1}{1-(-1)}} = v^{1/2} \quad \frac{dy}{dx} = \frac{1}{2} v^{-1/2} \frac{dv}{dx}$

$\frac{1}{2} v^{-1/2} \frac{dv}{dx} - \frac{1}{4x} v^{1/2} = x v^{-1/2}$ // multiply by $2v^{1/2}$

$\frac{dv}{dx} - \frac{1}{2x} v = 2x$ // lineare

$\mu(x) = e^{-\frac{1}{2}\int \frac{dx}{x}} = e^{-1/2 \ln|x|} = |x|^{-1/2} = x^{-1/2} \quad x>0$

$x^{-1/2} \frac{dv}{dx} - \frac{1}{2} x^{-3/2} v = 2x^{1/2}$

$\frac{d}{dx}(x^{-1/2} v) = 2x^{1/2}$

$x^{-1/2} v = \int 2x^{1/2} dx = \frac{4}{3} x^{3/2} + C$

$v = \frac{4}{3} x^2 + C\sqrt{x}$

$y = (4/3 x^2 + C\sqrt{x})^{1/2}$ plug $x=1, y=2$

$2 = (4/3 + C)^{1/2} \quad 4/3 + C = 4 \quad C = 4 - 4/3 = 8/3$

$$\boxed{y = \sqrt{4/3 x^2 + \frac{8}{3}\sqrt{x}}}$$

(10) **5.** Find all solutions to the equation

$$x\frac{dy}{dx} + 2y = \frac{1}{x}$$

divide both sides by x:

$$\frac{dy}{dx} + \frac{2}{x}y = \frac{1}{x^2}$$, <u>lineare</u>

integreating factor $\mu(x) = e^{\int \frac{2}{x}dx} = e^{2\ln|x|} = x^2$

$$x^2\frac{dy}{dx} + 2xy = 1$$

$$\frac{d}{dx}(x^2y) = 1$$

$$x^2y = \int 1\,dx = x + C$$

$$y = \frac{x+C}{x^2} = \frac{1}{x^2} + \frac{C}{x}$$

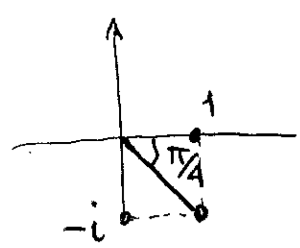
$$\boxed{y = \frac{1}{x^2} + \frac{C}{x}}$$

(10) 6. Write the expression in the form $a + bi$, where a and b are real numbers:

(a) $\frac{2-5i}{1-3i} = \frac{(2-5i)(1+3i)}{(1-3i)(1+3i)} = \frac{2-5i+6i+15}{1^2+3^2} = \frac{17+i}{10} =$

$$\boxed{= \frac{17}{10} + \frac{1}{10} i}$$

(b) $(1-i)^{20}$ (Hint: use exponential form)



$|1-i| = \sqrt{1^2+(-1)^2} = \sqrt{2}$

$\mathrm{Arg}(1-i) = -\frac{\pi}{4}$

$1-i = \sqrt{2} e^{-i\pi/4}$

$(1-i)^{20} = \left(\sqrt{2} e^{-i\frac{\pi}{4}}\right)^{20} = 2^{10} e^{-i5\pi} = 2^{10}(\cos(-5\pi) + i\sin(-5\pi)) =$

$= 2^{10}(-1 + i\cdot 0) = \boxed{-2^{10}}$

(10) 7.

(a) Express in the form $A\cos(\omega x - \phi)$:

$$\sin(3x) - \sqrt{3}\cos(3x) =$$

$$= \mathrm{Re}\left(-ie^{i3x}\right) - \sqrt{3}\,\mathrm{Re}\left(e^{i3x}\right) =$$

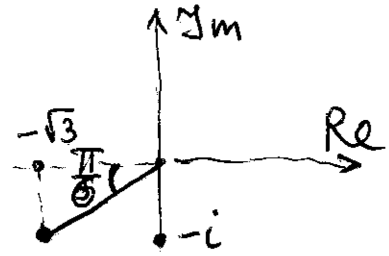
$$= \mathrm{Re}\left(\left(-i-\sqrt{3}\right)e^{i3x}\right) =$$

$$= \mathrm{Re}\left(2e^{i\frac{7\pi}{6}}e^{i3x}\right) =$$

$$= \mathrm{Re}\left(2e^{i\left(\frac{7\pi}{6}+3x\right)}\right) =$$

$$\boxed{= 2\cos\left(3x + \frac{7\pi}{6}\right)}$$

$$|-i-\sqrt{3}| = \sqrt{(-1)^2 + (-\sqrt{3})^2} = 2$$



$$\mathrm{Arg}(-i-\sqrt{3}) = \pi + \frac{\pi}{6} = \frac{7\pi}{6}$$

// Note: any other angle of the form $\frac{7\pi}{6} + 2\pi n$, for example, $-\frac{5\pi}{6}$, will work

(b) Express in the form $a\cos(\omega x) + b\sin(\omega x)$:

$$\sqrt{2}\sin(2x + \frac{\pi}{4}) =$$

$$= \sqrt{2}\left(\sin(2x)\cos\frac{\pi}{4} + \cos(2x)\sin\frac{\pi}{4}\right) =$$

$$= \sqrt{2}\left(\sin(2x)\frac{\sqrt{2}}{2} + \cos(2x)\frac{\sqrt{2}}{2}\right) =$$

$$\boxed{= \sin(2x) + \cos(2x)}$$

// Note: solution using complex numbers is also accepted.

(15) 8. Suppose that population of bisons in Konza Prairie satisfies the differential equation

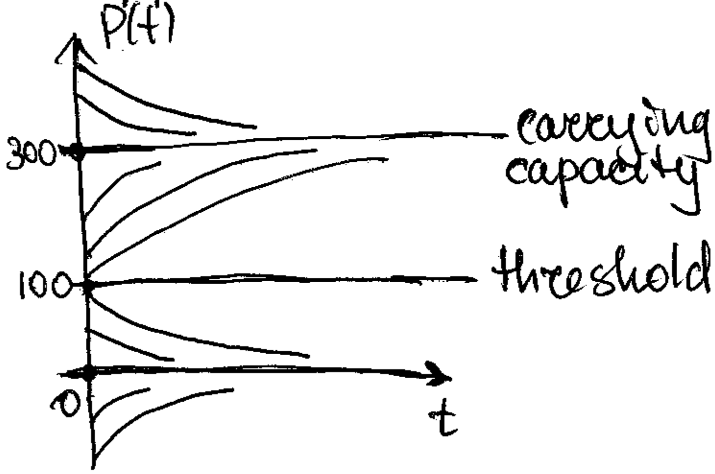
$$\frac{dP}{dt} = -0.1 * P(P-300)(P-100)$$

(a) Find the equilibrium points of the equation. Sketch several trajectories, including equilibrium solutions.

$$-0.1 \cdot P(P-300)(P-100) = 0$$

Eq. points are 0, 100, 300

P	$(-\infty;0)$	$(0;100)$	$(100,300)$	$(300,+\infty)$
$f(P)$	+	−	+	−



(b) Assume current population is equal to 85 bisons. Are they in danger of extinction? How many bisons do you need to buy to prevent extinction? Explain.

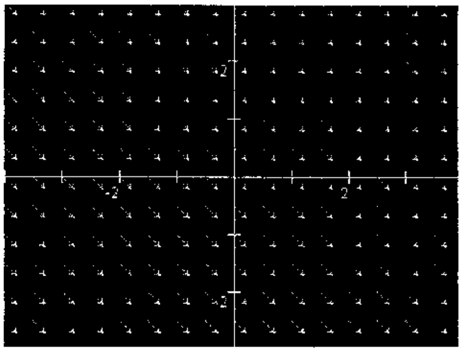
Yes, 85 is below threshold. You need to buy at least $100-85=15$ bisons to prevent extinction.

(c) Assume after the population had reached 275 bisons, additional 40 bisons were donated to Konza Prairie. Is that wise to accept the donation? Explain why/why not.

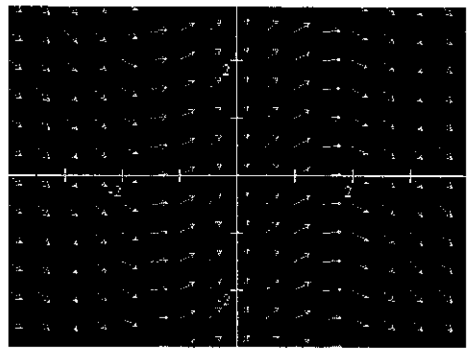
The total number will be $275+40=315$ bisons, which is more than carrying capacity. You should not accept donation since 15 bisons will eventually die.

(10) 9. Match the slope fields to the equations.

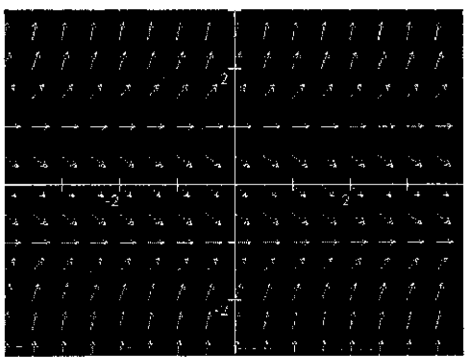
(a) $y' = -1$ (b) $y' = xy$ (c) $y' = -xy$ (d) $y' = y^2 - 1$ (e) $y' = cos(x)$



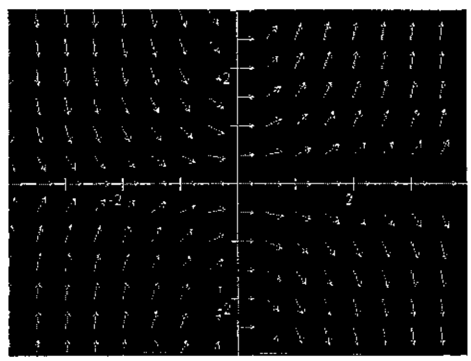
This is the slope field of equation a



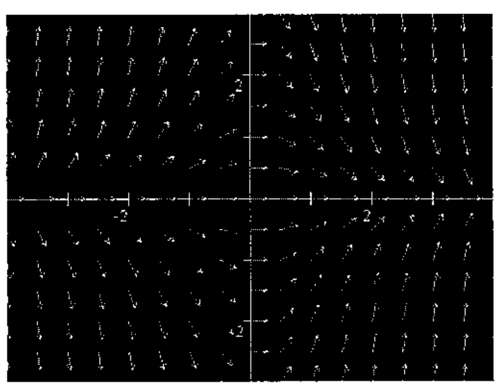
This is the slope field of equation e



This is the slope field of equation d



This is the slope field of equation b



This is the slope field of equation c