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Differential Equations - MATH 240, Spring 2007
Final Exam, May 9, 2007

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points								
page	8	9	10	11	12	13	14	total
points								

Show all your work in the space provided under each question. Each problem is worth 10 points.

1. Find the general solution to

$$\frac{1}{x}\frac{dy}{dx} - \frac{2y}{x^2} = x\cos x$$

this is linear $\frac{dy}{dx} - \frac{2}{x}y = x^2\cos x$

integ. factor $e^{\int -\frac{2}{x}dx} = \left(e^{\log x}\right)^{-2} = \frac{1}{x^2}$

$$\frac{1}{x^2}\frac{dy}{dx} - \frac{2}{x^3}y = \cos x$$

$$\frac{d}{dx}\left(\frac{1}{x^2}y\right) = \cos x$$

$$\int \frac{d}{dx}\left(\frac{1}{x^2}y\right)dx = \int \cos x\,dx$$

$$\frac{1}{x^2}y = \sin x + C$$

$$y = x^2\sin x + Cx^2$$

2. Use the improved Euler method with stepsize 1 to estimate $y(2)$ for the initial value problem.

$$y' = xy \qquad y(0) = 4$$

Here $h=1$, $x_0=0$, $x_1=1$, $x_2=2$ $\quad y_0=4$ $\quad f(x,y)=xy$

$\tilde{y}_1 = y_0 + hf(x_0,y_0) = 4 + 1(0\cdot 4) = 4$

$y_1 = y_0 + \frac{h}{2}[f(x_0,y_0) + f(x_1,\tilde{y}_1)]$

$= 4 + \frac{1}{2}[0\cdot 4 + 1\cdot 4] = 6$

$\tilde{y}_2 = y_1 + hf(x_1,y_1) = 6 + 1(1\cdot 6) = 12$

$y_2 = y_1 + \frac{h}{2}[f(x_1,y_1) + f(x_2,\tilde{y}_2)]$

$= 6 + \frac{1}{2}[1\cdot 6 + 2\cdot 12]$

$= 6 + \frac{1}{2}30$

$= 21$

$y(2) \approx 21$

3. The number $P(t)$ (measured in thousands) of pigeons in a city is described by the equation

$$\frac{dP}{dt} = -P(P-15)(P-4)$$

(a) Find the equilibrium points.

$P=0, 15, 4$

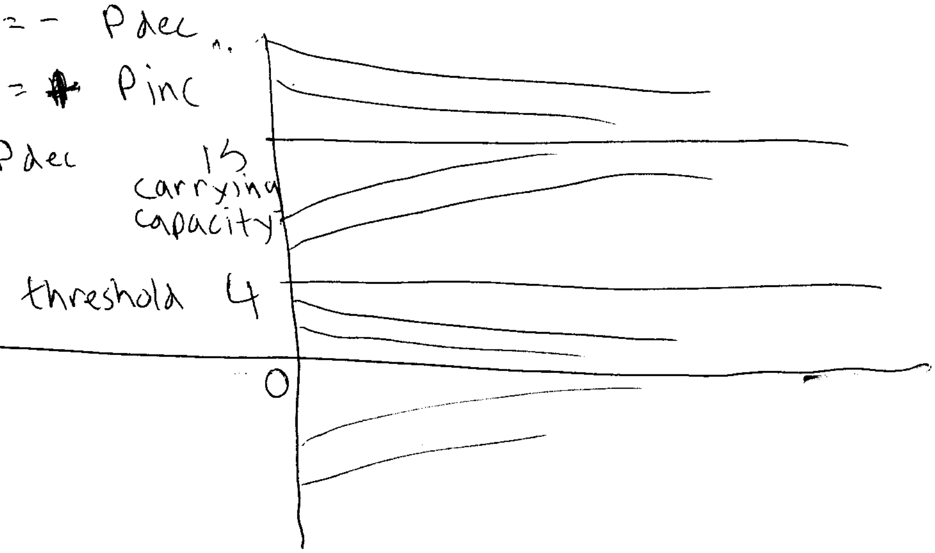
(b) Draw several trajectories. Include in your graph the critical points and label the threshold and carrying capacity.

if $P<0$ $\frac{dP}{dt} = -P(P-15)(P-4) = -(-)(-)(-)$ P inc

if $0<P<4$ $\frac{dP}{dt} = -(+)(-)(-) = -$ P dec

if $4<P<15$ $\frac{dP}{dt} = -(+)(+)(-) = +$ P inc

if $P>15$ $\frac{dP}{dt} = -(+)(+)(+)$ P dec



(c) Classify each critical point as stable or unstable.

0 is stable

4 is unstable

15 is stable

(d) In an attempt to eradicate the pigeon population, the city attempts to reduce their numbers to the level at which reproduction becomes difficult and the population declines. What is the maximum size population which will die out? Explain your answer. As long as you get the population below 4000, they will die out

4. Rewrite as a system and use the Euler method with stepsize $h = .5$ to estimate $x(1)$ for the initial value problem

$$x'' + x' + x^2 = 0 \quad x(0) = 0 \quad x'(0) = 1$$

$x' = y$

$y' + y + x^2 = 0$

so get $x' = y \qquad x(0) = 0$

$y' = -y - x^2 \qquad y(0) = 1$

Here $t_0 = 0$, $t_1 = .5$, $t_2 = 1$, $h = .5$, $x_0 = 0$, $y_0 = 1$

$x_1 = x_0 + h f(x_0, y_0) = 0 + .5(1) = .5$

$y_1 = y_0 + h g(x_0, y_0) = 1 + .5(-1 - 0^2) = .5$

$x_2 = x_1 + h f(x_1, y_1) = .5 + .5(.5) = .75$

5. Find the general solution.

$$y' = \frac{y + xe^{-\frac{y}{x}}}{x}.$$

This is homogeneous.

Let $y = xv$ then $\frac{dy}{dx} = v + x\frac{dv}{dx}$

equation becomes:

$$v + x\frac{dv}{dx} = \frac{xv + xe^{-v}}{x}$$

$$v + x\frac{dv}{dx} = v + e^{-v}$$

$$x\frac{dv}{dx} = e^{-v}$$

$$\int e^{v}\,dv = \int \frac{dx}{x}$$

$$e^{v} = \log|x| + C$$

$$e^{y/x} = \log|x| + C$$

6. Solve the initial value problem

$$x'' + 6x' + 5x = 2\delta(t-1) \qquad x(0) = -3, \quad x'(0) = 0.$$

$$s^2 X - (-3)s - 0 + 6(sX - (-3)) + 5X = 2e^{-s}$$

$$s^2 X + 3s + 6sX + 18 + 5X = 2e^{-s}$$

$$(s^2 + 6s + 5)X = -3s - 18 + 2e^{-3}$$

$$X = \frac{-3s-18}{(s+5)(s+1)} + \frac{2e^{-s}}{(s+5)(s+1)}$$

$$\frac{-3s-18}{(s+5)(s+1)} = \frac{A}{s+5} + \frac{B}{s+1}$$

$$-3s - 18 = A(s+1) + B(s+5)$$

$s=-1$ $\quad -15 = B \cdot 4$

$-\frac{15}{4} = B$

$s=-5$ $\quad -3 = A(-4)$

$\frac{3}{4} = A$

$$\frac{1}{(s+5)(s+1)} = \frac{C}{s+5} + \frac{D}{s+1}$$

$$1 = C(s+1) + D(s+5)$$

$s=-1$ $\quad 1 = D \cdot 4$

$\frac{1}{4} = D$

$s=-5$ $\quad 1 = -4C$

$-\frac{1}{4} = C$

so $X = \frac{3}{4}\frac{1}{s+5} - \frac{15}{4}\frac{1}{s+1} + -\frac{1}{2}e^{-s}\frac{1}{s+5} + \frac{1}{2}e^{-s}\frac{1}{s+1}$

$$x = \frac{3}{4}e^{-5t} - \frac{15}{4}e^{-t} + -\frac{1}{2}u(t-1)e^{-5(t-1)} + \frac{1}{2}e^{-(t-1)}u(t-1)$$

7. Find the general solution.

$$x^2y'' - xy' - 8y = x^2 \qquad \text{(assume } x > 0\text{)}.$$

$x^2y'' - xy' - 8y = 0$

$r(r-1) - r - 8 = 0$

$r^2 - 2r - 8 = 0$

$(r-4)(r+2) = 0 \qquad r = 4, -2$

$y_h = c_1x^4 + c_2x^{-2}$

To find y_p use variation of parameters.

eq'n is $y'' - \frac{1}{x}y' - \frac{8}{x^2}y = 1$

$W(x^4, x^{-2}) = \det\begin{bmatrix} x^4 & x^{-2} \\ 4x^3 & -2x^{-3} \end{bmatrix} = -2x - 4x = -6x$

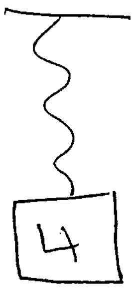
$y_p = x^4\int \frac{-x^{-2}\cdot 1}{-6x}dx + x^{-2}\int \frac{x^4 \cdot 1}{-6x}dx$

$= \frac{x^4}{6}\int \frac{1}{x^3}dx + \frac{1}{x^2}\left(-\frac{1}{6}\right)\int x^3 dx$

$= \frac{x^4}{6}\left(-\frac{1}{2x^2}\right) + -\frac{1}{6x^2}\frac{x^4}{4} = -\frac{1}{12}x^2 - \frac{1}{24}x^2 = -\frac{1}{8}x^2$

So $y = c_1x^4 + c_2x^{-2} - \frac{1}{8}x^2$

8. A mass of $4\,kg$ is attached to a spring which causes the spring to stretch $9.8\,m$. After coming to equilibrium, the mass is given an initial displacement of $1\,m$. There is no damping. What initial velocity should be given to the mass so that the amplitude of the resulting motion is $4\,m$? Take $g = 9.8\,\frac{m}{sec^2}$.



$$mg = k\Delta l$$
$$4(9.8) = k(9.8)$$
$$4 = k$$

$$4x'' + 4x = 0 \qquad x(0) = 1, \; x'(0) = ?$$

use energy.

$$\text{Initially } E = \tfrac{1}{2} m\, x'(0)^2 + \tfrac{1}{2} k\, x(0)^2$$
$$= \tfrac{1}{2}\cdot 4\, x'(0)^2 + \tfrac{1}{2}\cdot 4 \cdot 1$$
$$= 2x'(0)^2 + 2$$

At max. amplitude all energy is potential

$$E = \tfrac{1}{2} k A^2 = \tfrac{1}{2}\cdot 4 \cdot 4^2 = 32$$

E is conserved so

$$32 = 2x'(0)^2 + 2$$
$$15 = x'(0)^2$$

$$\pm\sqrt{15}\,\frac{m}{sec} = x'(0)$$

9. Solve $y'' + 2xy = 0$, $y(0) = 0$, $y'(0) = 1$ using power series. Find the recurrence relation and the first three nonzero terms of the series solution.

$y = \sum_{n=0}^{\infty} a_n x^n \qquad y' = \sum_{n=1}^{\infty} n a_n x^{n-1} \qquad y'' = \sum_{n=2}^{\infty} (n-1)n\, a_n x^{n-2}$

$$\sum_{n=2}^{\infty} (n-1)n\, a_n x^{n-2} + \sum_{n=0}^{\infty} 2a_n x^{n+1} = 0$$

$j = n-2 \qquad\qquad j = n+1$

$$\sum_{j=0}^{\infty} (j+1)(j+2) a_{j+2} x^j + \sum_{j=1}^{\infty} 2a_{j-1} x^j = 0$$

$$2a_2 + \sum_{j=1}^{\infty} \left((j+1)(j+2)a_{j+2} + 2a_{j-1}\right) x^j = 0$$

$a_2 = 0 \qquad (j+1)(j+2)a_{j+2} + 2a_{j-1} = 0$

$$a_{j+2} = \frac{-2a_{j-1}}{(j+1)(j+2)}$$

recurrence relation

now $a_0 = 0,\ a_1 = 1,\ a_2 = 0,\ a_3 = \frac{-2}{2\cdot 3} a_0 = 0,$

$a_4 = \frac{-2a_1}{3\cdot 4} = -\frac{1}{6}, \quad a_5 = 0,\ a_6 = 0,$

$a_7 = \frac{-2a_4}{6\cdot 7} = -\frac{1}{21}\left(-\frac{1}{6}\right) = \frac{1}{126}, \ldots$

$$y = x - \frac{1}{6}x^4 + \frac{1}{126}x^7 + \ldots$$

10. Consider the differential equation

$$2(x-2)^2 y'' + 3xy' + (x-2)y = 0.$$

(a) Find the ordinary points and the singular points of this differential equation.

$$y'' + \frac{3x}{2(x-2)^2} y' + \frac{1}{2(x-2)} y = 0$$

$x = 2$ is the only singular point.

All other points are ordinary points.

(b) Classify each singular point you found in (a) as a regular singular point or an irregular singular point.

$$x-2 \; \frac{3x}{2(x-2)^2} = \frac{3x}{2(x-2)} \quad \text{not analytic at } x=2$$

$$(x-2)^2 \; \frac{1}{2(x-2)} = \frac{x-2}{2}$$

so 2 is an irregular singular point

(c) It is desired to solve this equation using a power series centered at $x = 10$. Give a lower bound for the radius of convergence of such a power series solution.

radius of converg. is distance from 10 to 2

so lower bound for radius of convergence is 8.

11. Find a Frobenius series solution corresponding to the larger root of the indicial equation of the differential equation

$$2x^2y'' - xy' + xy = 0.$$

Give at least three nonzero terms in the series.

$$y'' - \frac{x}{2x^2}y' + \frac{x}{2x^2}y = 0 \qquad \lim_{x\to 0} x\left(-\frac{1}{2x}\right) = -\frac{1}{2}$$

$$y'' - \frac{1}{2x}y' + \frac{x}{2}y = 0 \qquad \lim_{x\to 0} x\left(\frac{x}{2}\right) = 0$$

indicial eq'n is $r(r-1) - \frac{1}{2}r = 0$

$r(r-\frac{3}{2}) = 0 \qquad r = 0, \frac{3}{2} \qquad$ larger root is $\frac{3}{2}$

$$y = \sum_{n=0}^{\infty} a_n x^{n+\frac{3}{2}} \quad y' = \sum_{n=0}^{\infty} a_n(n+\tfrac{3}{2})x^{n+\frac{1}{2}}, \quad y'' = \sum_{n=0}^{\infty} a_n(n+\tfrac{3}{2})(n+\tfrac{1}{2})x^{n-\frac{1}{2}}$$

so feeding in, we get:

$$\sum_{n=0}^{\infty} 2a_n(n+\tfrac{3}{2})(n+\tfrac{1}{2})x^{n+\frac{3}{2}} - \sum_{n=0}^{\infty} a_n(n+\tfrac{3}{2})x^{n+\frac{3}{2}} + \sum_{n=0}^{\infty} a_n x^{n+\frac{5}{2}} = 0$$

$n=j \qquad\qquad n=j \qquad\qquad j=n+1$

$$\sum_{j=0}^{\infty} 2a_j(j+\tfrac{3}{2})(j+\tfrac{1}{2})x^{j+\frac{3}{2}} - \sum_{j=0}^{\infty} a_j(j+\tfrac{3}{2})x^{j+\frac{3}{2}} + \sum_{j=1}^{\infty} a_{j-1}x^{j+\frac{3}{2}} = 0$$

$$2a_0(\tfrac{3}{2})(\tfrac{1}{2})x^{\frac{3}{2}} - a_0(\tfrac{3}{2})x^{\frac{3}{2}} + \sum_{j=1}^{\infty}\left(2a_j(j+\tfrac{3}{2})(j+\tfrac{1}{2}) - a_j(j+\tfrac{3}{2}) + a_{j-1}\right)x^{j+\frac{3}{2}} = 0$$

so $2a_0(\frac{3}{2})(\frac{1}{2}) - \frac{3}{2}a_0 = 0$, that is $a_0 =$ anything.. Say $a_0 = 1$

Also $\quad a_j = \dfrac{-a_{j-1}}{2(j+\frac{3}{2})(j+\frac{1}{2}) - (j+\frac{3}{2})} \qquad a_1 = \dfrac{-a_0}{2(\frac{5}{2})(\frac{3}{2}) - \frac{5}{2}} = \dfrac{-1}{\frac{10}{2}}$

$= -\frac{1}{5}$

$$a_2 = \frac{-a_1}{2\frac{7}{2}\cdot\frac{5}{2} - \frac{7}{2}} = +\frac{1}{5}\,\frac{1}{\frac{28}{2}} = \frac{1}{5}\cdot\frac{1}{14} = \frac{1}{70}$$

$$y = 1x^{\frac{3}{2}} - \frac{1}{5}x^{\frac{5}{2}} + \frac{1}{70}x^{\frac{7}{2}} + \ldots$$

12. (a) Find the Taylor polynomial of $\frac{1}{1+2x}$ around $x = 0$. What is its radius of convergence?

$$\frac{1}{1+2x} = \frac{1}{1-(-2x)} = 1 - 2x + (-2x)^2 + (-2x)^3 + \cdots$$

$$= 1 - 2x + 2^2x^2 - 2^3x^3 + 2^4x^4 - \cdots$$

valid for $|-2x| < 1$

$|x| < \frac{1}{2}$

radius of convergence is $\frac{1}{2}$.

(b) Find the Taylor series for $(1+x)e^x$ about $x = 0$.

$$(1+x)e^x = (1+x)\left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \cdots\right)$$

$$= 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots$$

$$+ x + x^2 + \frac{x^3}{2!} + \frac{x^4}{3!} + \frac{x^5}{4!} + \cdots$$

$$= 1 + 2x + \left(1 + \frac{1}{2!}\right)x^2 + \left(\frac{1}{2!} + \frac{1}{3!}\right)x^3 + \cdots$$

13. Consider the differential equation

$$mx'' + cx' + kx = 0 \qquad x(0) = 0 \;\; x'(0) = 1$$

which describes a mass-spring-damping system. Here m,c, and k are all positive.

(a) Suppose that the system is underdamped. What can you say about the location of the poles of $\mathcal{L}$ in the complex plane?

$$ms^2 X - 1 + csX + kX = 0$$

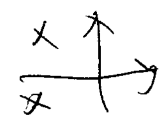
$$(ms^2 + cs + k) X = 1$$

$$X = \frac{1}{ms^2 + cs + k}$$

poles are at $s = \frac{-c}{2m} \pm \frac{\sqrt{c^2 - 4km}}{2m}$

underdamped means $c^2 - 4km < 0$

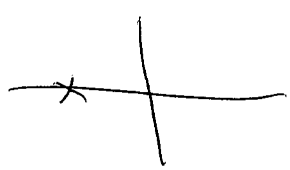
get 2 complex conjugate poles in left half plane



(b) Suppose that the system is critically damped. What can you say about the location of the poles of $\mathcal{L}$ in the complex plane?

critically damped means $c^2 - 4km = 0$

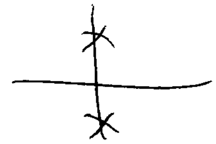
get one real root $s = \frac{-c}{2m} < 0$



(c) Suppose now that $c = 0$. What can you say about the location of the poles of $\mathcal{L}$ in the complex plane?

roots are $s = \frac{-c}{2m} \pm \frac{\sqrt{-4km}}{2m}$

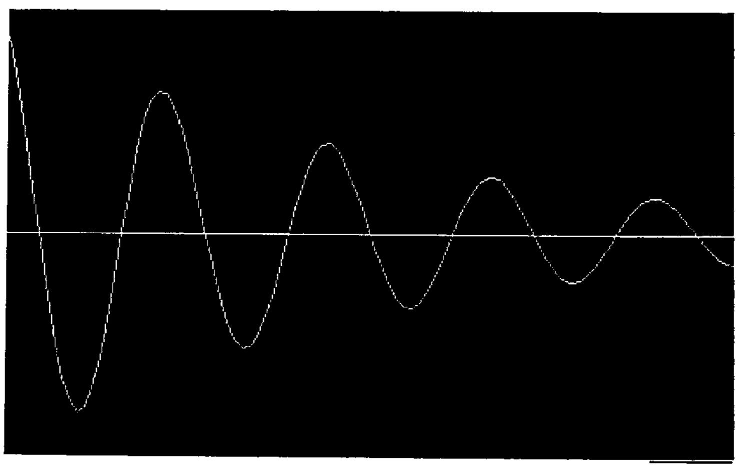
pure imaginary conjugates



14. (This problem is worth 20 points.) Match the graphs to the equations. In each graph the horizontal axis ranges from 0 to 20 and the vertical axis ranges from -10 to 10.

1. $x'' + x = \cos t$	$x(0) = 0$	$x'(0) = 0$	Its graph is B
2. $x'' + 16x = 0$	$x(0) = 0$	$x'(0) = 30$	Its graph is H
3. $x'' + x = 0$	$x(0) = 10$	$x'(0) = 0$	Its graph is G
4. $5x'' + x' + 10x = 0$	$x(0) = 10$	$x'(0) = 0$	Its graph is A
5. $x'' + x = 0$	$x(0) = 0$	$x'(0) = 9$	Its graph is D
6. $5x'' + x' + 10x = 0$	$x(0) = 0$	$x'(0) = 10$	Its graph is F
7. $x'' + 4x' + 3x = 0$	$x(0) = 10$	$x'(0) = 0$	Its graph is E
8. $x'' + 49x = 10\cos 7t$	$x(0) = 0$	$x'(0) = 0$	Its graph is C
9. $x'' + 10x' + 3x = 0$	$x(0) = 0$	$x'(0) = 100$	Its graph is J
10. $x'' + 400x = 200\cos 18t$	$x(0) = 0$	$x'(0) = 0$	Its graph is I

A



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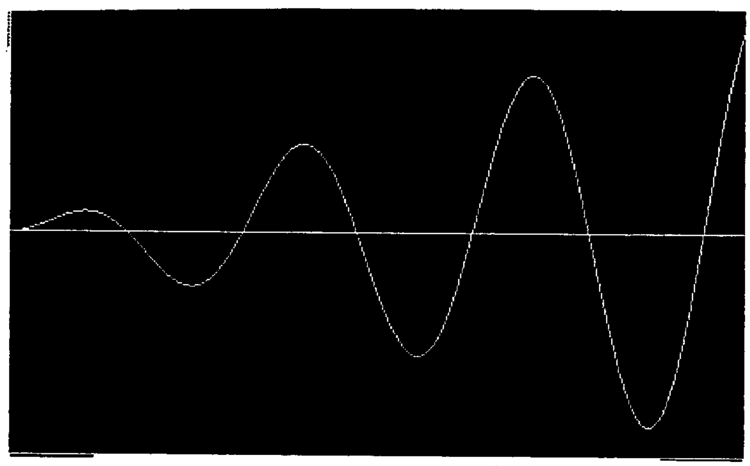

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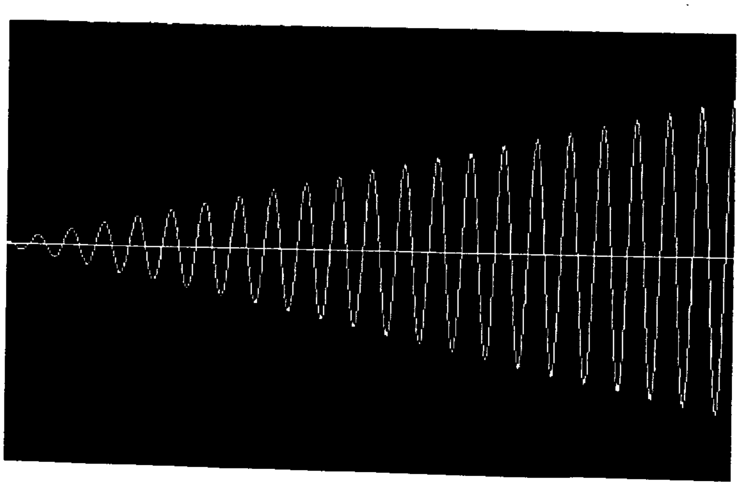

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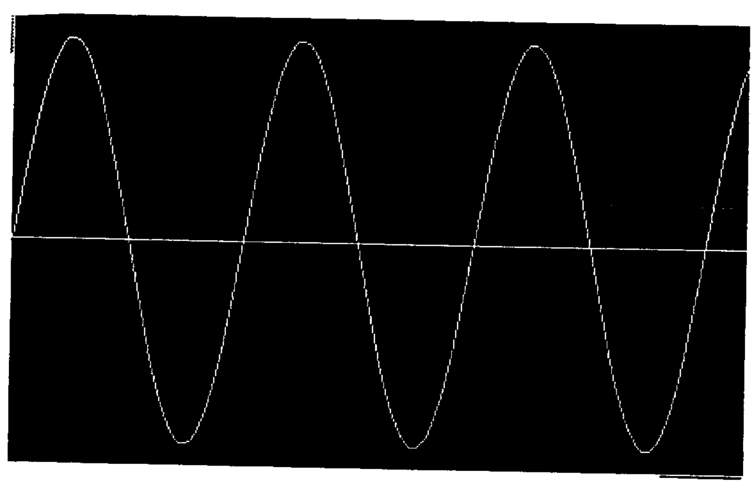

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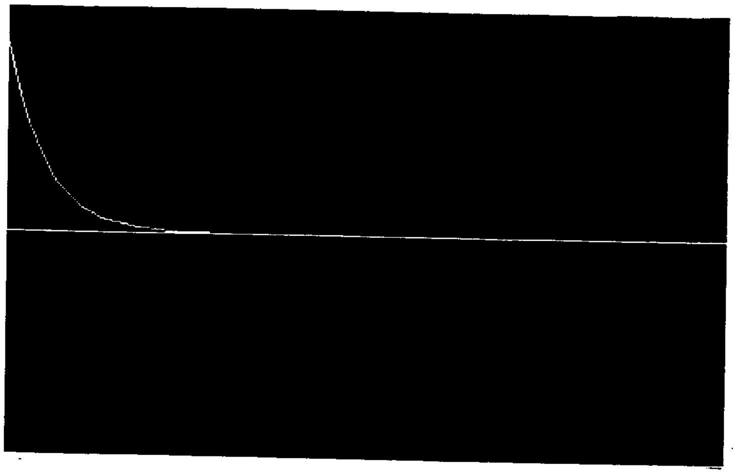

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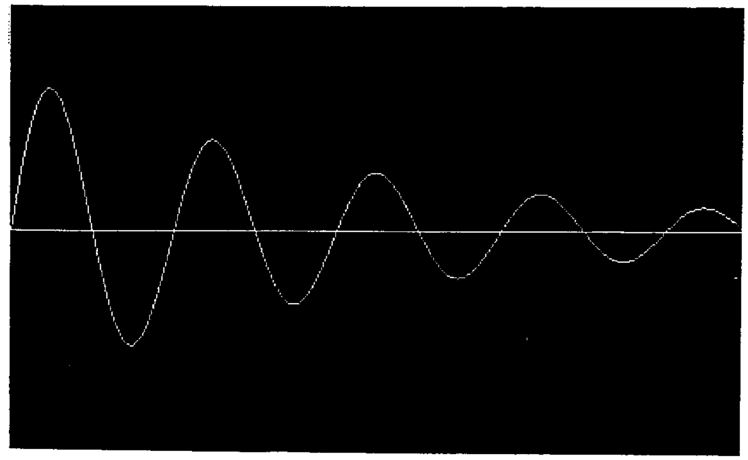

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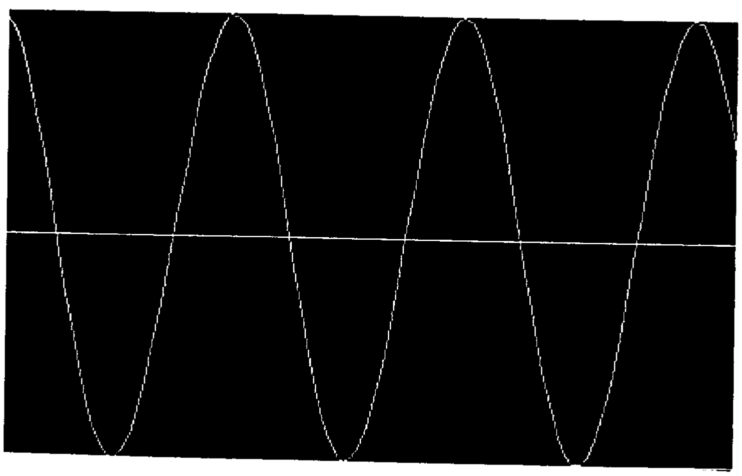
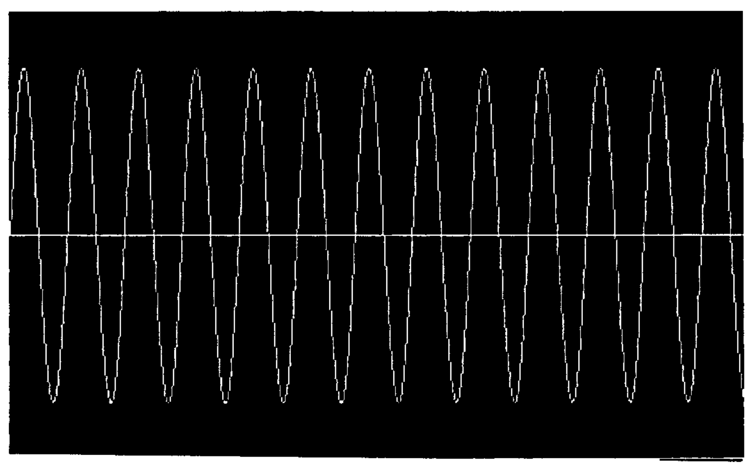


H



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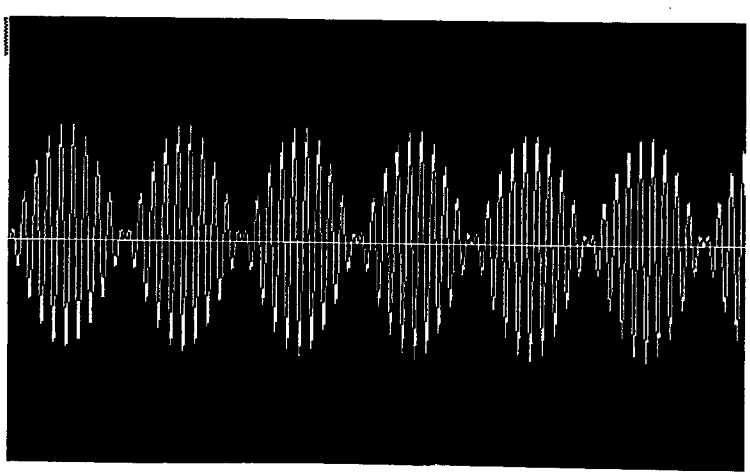

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