Name:______________________________

1. Solve the initial value problem $x^2\frac{dy}{dx}+2xy-1=0, \quad y(1)=2$.

Divide by x^2 $\quad \frac{dy}{dx}+\frac{2}{x}y=\frac{1}{x^2}$ $\quad$ Linear

Step 1 $\quad \mu = e^{\int \frac{2}{x}dx} = e^{2\log x + \cancel{c}^{\to 0}} = x^2$

Step 2 $\quad \underbrace{x^2\frac{dy}{dx}+2xy}=1$

Step 3 $\quad \frac{d}{dx}(x^2y)=1$

Step 4 $\quad x^2y=\int 1\,dx = x+C$

Step 5 $\quad y=\frac{x+C}{x^2}$

Now $y(1)=2$

So

$2=\frac{1+C}{1}$

$C=1$

$$\boxed{y=\frac{x+1}{x^2}}$$

Alternatively

$$\underbrace{x^2}_{N}\frac{dy}{dx}+\underbrace{(2xy-1)}_{M}=0$$

$\frac{\partial M}{\partial y}=2x=\frac{\partial N}{\partial x}$ so the equation is Exact

$F=\int 2xy-1\,\partial x = x^2y-x+C(y)$

$F=\int x^2\,\partial y = x^2y+\tilde{C}(x)+0$

$F=x^2y-x$

$$x^2y-x=K$$

Now $y(1)=2$ $\quad 1\cdot 2-1=K$

$1=K$

$$x^2y-x=1$$

$$x^2y=x+1$$

$$\boxed{y=\frac{x+1}{x^2}}$$

Name:________________________________

2. Find the general solution to $\frac{d^2y}{dx^2}+6\frac{dy}{dx}+10y=2\cos(x)$.

Find Homo Soln

$$(D^2+6D+10)y_h=0$$

roots $-3\pm i$

$$y_h=c_1e^{-3x}\cos(x)+c_2e^{-3x}\sin(x)$$

Using Real Techniques

$$y_p=A\cos(x)+B\sin(x)$$
$$y_p'=-A\sin(x)+B\cos(x)$$
$$y_p''=-A\cos(x)-B\sin(x)$$

$$y_p''+6y_p'+10y_p$$
$$=(-A+6B+10A)\cos(x)+(-B-6A+10B)\sin(x)$$
$$=(9A+6B)\cos(x)+(-6A+9B)\sin(x)$$
$$\stackrel{\text{set}}{=}2\cos(x)$$

Equating Coefficients

$$9A+6B=2 \xrightarrow{\times 2} 18A+12B=4$$
$$-6A+9B=0 \xrightarrow{\times 3} -18A+27B=0$$
$$39B=4$$
$$B=4/39$$
$$A=6/39$$

$$y_p=6/39\cos(x)+4/39\sin(x)$$

Using Complex Techniques

$$2\cos(x)=\mathrm{Re}[2e^{ix}]$$
$$\tilde{y}_p=Ae^{ix}$$
$$\tilde{y}_p'=iAe^{ix}$$
$$\tilde{y}_p''=-Ae^{ix}$$
$$\tilde{y}_p''+6\tilde{y}_p'+10\tilde{y}_p=(-A+6iA+10A)e^{ix}$$
$$=(9+6i)Ae^{ix}$$
$$\stackrel{\text{set}}{=}2e^{ix}$$

$$(9+6i)A=2$$
$$A=\frac{2}{9+6i}\cdot\frac{9-6i}{9-6i}$$
$$=\frac{18-12i}{117}=\frac{6}{39}-\frac{4}{39}i$$

$$y_p=\mathrm{Re}[\tilde{y}_p]=\mathrm{Re}\left[\left(\frac{6}{39}-\frac{4}{39}i\right)(\cos(x)+i\sin(x))\right]$$
$$=\frac{6}{39}\cos(x)+\frac{4}{39}\sin(x)$$

$$\boxed{y=\frac{6}{39}\cos(x)+4/39\sin(x)+c_1e^{-3x}\cos(x)+c_2e^{-3x}\sin(x)}$$

Name:______________________________

3. Solve the initial value problem $\frac{d^2y}{dx^2}+(x+1)y=0, \quad y(0)=2, y'(0)=3$.

Variable Coeff $\to$ Series Soln.

Step 1 $y=\sum_{n=0}^{\infty} a_n x^n \to (x+1)y=\sum_{n=0}^{\infty} a_n x^{n+1}+\sum_{n=0}^{\infty} a_n x^n$

$y'=\sum_{n=1}^{\infty} n a_n x^{n-1}$

$y''=\sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} \to y''=\sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}$

Step 2 $(x+1)y=\sum_{j=1}^{\infty} a_{j-1}x^j+\sum_{n=0}^{\infty} a_n x^n$

$j=n+1$
$n=j-1$

$y''=\sum_{k=0}^{\infty}(k+2)(k+1)a_{k+2}x^k$

$k=n-2$
$n=k+2$

Step 3 $y''+(x+1)y=\sum_{m=0}^{\infty}(m+2)(m+1)a_{m+2}x^m+\sum_{m=0}^{\infty} a_{m-1}x^m+\sum_{m=0}^{\infty} a_m x^m=0$

Step 4 $(2a_2+a_0)+\sum_{m=1}^{\infty}\left[(m+2)(m+1)a_{m+2}+a_{m-1}+a_m\right]x^m=0$

Step 5 $2a_2+a_0=0 \to a_2=-\frac{1}{2}a_0$

$(m\geq 1)$

$(m+2)(m+1)a_{m+2}+a_{m-1}+a_m=0 \to a_{m+2}=-\frac{a_m+a_{m-1}}{(m+2)(m+1)}$

Step 6

$a_0=2$

$a_1=3$

$a_2=-\frac{1}{2}\cdot 2=-1$

$a_3=-\frac{2+3}{3\cdot 2}=-\frac{5}{6}$

$a_4=-\frac{-1+3}{4\cdot 3}=-\frac{1}{6}$

$\vdots$

$$\boxed{y(x)=2+3x-x^2-\frac{5}{6}x^3-\frac{1}{6}x^4+\dots}$$

Name:________________________________

4. Find the general solution to $y''-e^x y'+(e^x-1)y=0$. Hint: make the substitution $y=e^x u$ followed later by the substitution $v=u'$. Your final answer will have an integral you can't simplify.

$$y = e^x u$$
$$y' = e^x u + e^x u'$$
$$y'' = e^x u + e^x u' + e^x u' + e^x u''$$
$$= e^x u + 2e^x u' + e^x u''$$

(Applying the product rule repeatedly)

$$y'' - e^x y' + (e^x-1)y = e^x u + 2e^x u' + e^x u'' - e^x(e^x u + e^x u')$$
$$+ (e^x - 1)e^x u$$
$$= \cancel{e^x u} + 2e^x u' + e^x u'' - \cancel{e^{2x} u} - e^{2x} u'$$
$$+ \cancel{e^{2x} u} - \cancel{e^x u}$$
$$= e^x u'' + (2e^x - e^{2x})u' = 0$$

Divide through by e^x

$$u'' + (2 - e^x)u' = 0$$

Let $v = u'$

$\frac{dv}{dx} = u''$

$$\frac{dv}{dx} + (2 - e^x)v = 0 \qquad \underline{\text{Separable}}$$
$$\frac{dv}{dx} = (e^x - 2)v$$
$$\frac{dv}{v} = (e^x - 2)dx$$
$$\log v = e^x - 2x + C$$
$$v = ke^{e^x - 2x}$$

Now Back Substitute

$$u' = ke^{e^x - 2x}$$
$$u = k\int_0^x e^{e^t - 2t}\,dt + C$$
$$\boxed{y = e^x u = ke^x \int_0^x e^{e^t - 2t}\,dt + Ce^x}$$

Name:_______________________________

5. Solve the initial value problem $\begin{aligned} \frac{dx}{dt} &= 2x - 6y, \quad x(0) = 2 \\ \frac{dy}{dt} &= x + 7y, \quad y(0) = 1 \end{aligned}$.

Take Laplace transforms

$$\begin{bmatrix} sX - 2 = 2X - 6Y \\ sY - 1 = X + 7Y \end{bmatrix} \rightarrow \begin{bmatrix} (s-2)X + 6Y = 2 \\ -X + (s-7)Y = 1 \end{bmatrix}$$

$$(s-7)(s-2)X + 6(s-7)Y = 2(s-7)$$
$$+ \quad 6X - 6(s-7)Y = -6$$
$$[s^2 - 9s + 14 + 6]X = 2s - 14 - 6$$
$$X = \frac{2s-20}{s^2-9s+20}$$

$$\frac{2s-20}{(s-4)(s-5)} = \frac{A}{s-4} + \frac{B}{s-5}$$
$$2s - 20 = A(s-5) + B(s-4)$$
$s=5$: $-10 = B$

$s=4$: $-12 = -A \rightarrow A = 12$

$$(s-2)X + 6Y = 2$$
$$+ \quad -(s-2)X + (s-2)(s-7)Y = s-2$$
$$(s^2 - 9s + 14 + 6)Y = s$$
$$Y = \frac{s}{s^2-9s+20}$$

$$\frac{s}{(s-4)(s-5)} = \frac{A}{(s-4)} + \frac{B}{s-5}$$
$$s = A(s-5) + B(s-4)$$
$s=5$: $5 = B$

$s=4$: $4 = -A \rightarrow A = -4$

$$\boxed{x(t) = 12e^{4t} - 10e^{5t}, \quad y(t) = -4e^{4t} + 5e^{5t}}$$

Name:________________________________

6. Find the ~~general~~ solution to $a_{n+2} - 5a_{n+1} + 4a_n = 2^n$, $a_0 = 0, a_1 = 0$.

Find Homo Soln

$a_n^{(h)} = \lambda^n$

$$\lambda^{n+2} - 5\lambda^{n+1} + 4\lambda^n = 0$$

$$\lambda^2 - 5\lambda + 4 = 0$$

$$(\lambda - 4)(\lambda - 1) = 0$$

$$\lambda = 4, \lambda = 1$$

$$a_n^{(h)} = C_1 4^n + C_2$$

Find Part. Soln

$a_n^{(p)} = A2^n$

$$\frac{A2^{n+2} - 5A2^{n+1} + 4A2^n}{2^n} = \frac{2^n}{2^n}$$

$$4A - 10A + 4A = 1$$

$$-2A = 1$$

$$A = -1/2$$

$$a_n^{(p)} = -\frac{1}{2}2^n$$

General Solution

$$a_n = -\frac{1}{2}2^n + C_1 4^n + C_2$$

Solve initial value problem

$$a_0 = -\frac{1}{2} + C_1 + C_2 \overset{\text{set}}{=} 0 \rightarrow C_1 + C_2 = 1/2$$

$$a_1 = -1 + 4C_1 + C_2 \overset{\text{set}}{=} 0 \rightarrow 4C_1 + C_2 = 1$$

Subtract 1st eqn from 2nd

$$3C_1 = 1/2$$

$$C_1 = 1/6$$

$$C_2 = 1/3$$

So $\boxed{a_n = -\frac{1}{2}2^n + \frac{1}{6}4^n + \frac{1}{3}}$

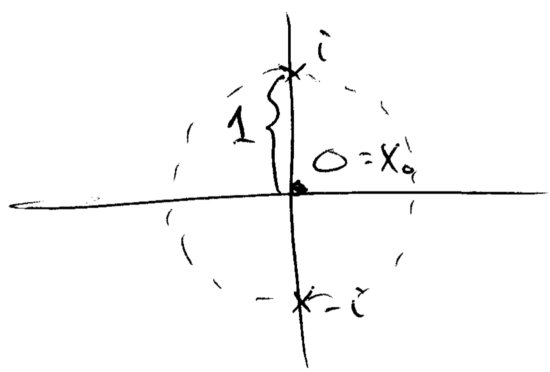
Name:________________________________

7. What is the "a priori" lower bound for the radius of convergence for the series solution about $x_0 = 0$ for $(x^2+1)y''+2xy'-y=0$?

The radius of convergence is at least as large as the distance to the nearest singular point

Singular points when $x^2+1=0$

$x = \pm i$



So the radius of convergence $\geq \boxed{1}$

Name:___________________________________

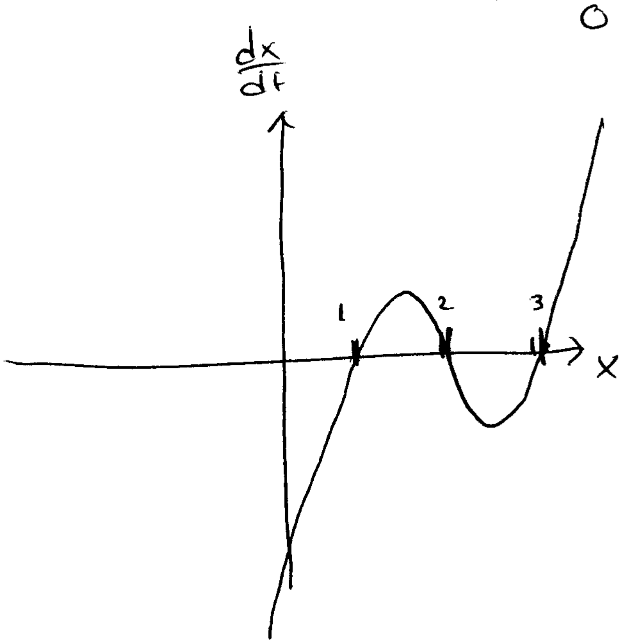
8. For what values of $x(0)$ will $\lim_{t\to\infty} x(t) = 2$ if $\frac{dx}{dt} = x^3 - 6x^2 + 11x - 6$?

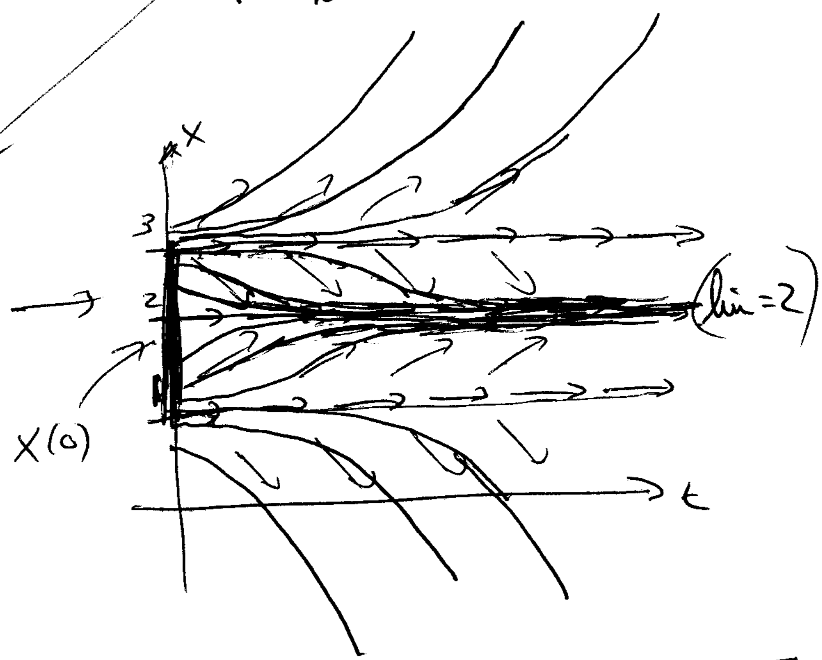
First Graph $\frac{dx}{dt}$ vs. x. We need the roots of $x^3 - 6x^2 + 11x - 6$.
You can use a graphing calculator or observe that $1^3 - 6\cdot 1^2 + 11\cdot 1 - 6 = 0$
So $x = 1$ is a root. Factoring this out gives

$$\begin{array}{r|l} & x^2 - 5x + 6 \\ \hline x-1 & x^3 - 6x^2 + 11x - 6 \\ & \underline{x^3 - x^2} \\ & \quad -5x^2 + 11x \\ & \quad \underline{-5x^2 + 5x} \\ & \qquad 6x - 6 \\ & \qquad \underline{6x - 6} \\ & \qquad\quad 0 \end{array}$$

So $x^3 - 6x^2 + 11x - 6 = (x-1)(x^2 - 5x + 6)$
$= (x-1)(x-2)(x-3)$

Now we graph the cubic curve and then use this to plot the x vs. t slope field





We observe $\lim_{t\to\infty} x(t) = 2$ when $\boxed{1 < x(0) < 3}$

Name:________________________________

9. Approximate $y(.2)$ using the improved Euler's method with step size 0.1 where $\frac{dy}{dx} = x - 2y, \quad y(0) = 1$.

$$\frac{dy}{dx} = f(x,y) = x - 2y$$

x	y	$f(x,y)$	$\tilde{x} = x+h$	$\tilde{y} = y+hf$	$f(\tilde{x},\tilde{y})$	$\frac{f+\tilde{f}}{2}$
0	1	-2	.1	.8	-1.5	-1.75
.1	.825	-1.55	.2	.67	-1.14	-1.345
.2	.6905					

So $\boxed{y(.2) \approx .6905}$

(Note: this equation is linear so you can solve to find the exact value is $y(.2) = 1.25e^{-.4} - .15 = .6879\ldots$, which we can use to check our work, but the problem specifically asks for the approximation, not the exact value).

Name:________________________________

10. A mass of 1kg is attached to a spring causing it to stretch .1m. The spring is then placed in free motion. If you don't know the damping constant, what is the range of possible values for the quasiperiod (the time for the spring to complete one cycle)? Use $g = 9.8$ m/sec^2.

$$mg = k\Delta l$$
$$1\text{kg} \cdot 9.8\,\text{m/sec}^2 = k \cdot .1\text{m}$$
$$k = 98\ \text{kg/sec}^2$$

So our equation becomes unknown damping

$$x'' + cx' + 98x = 0$$
$$(D^2 + cD + 98)x = 0$$

roots $-\frac{c}{2} \pm \frac{\sqrt{c^2 - 4\cdot 98}}{2} = -\frac{c}{2} \pm \frac{\sqrt{392 - c^2}}{2}\,i$

$= -\frac{c}{2} \pm \sqrt{\frac{392 - c^2}{2^2}}\,i$

$= -\frac{c}{2} \pm \sqrt{98 - (c/2)^2}\,i$

So $x(t) = Ae^{-\frac{c}{2}t}\cos\left(\sqrt{98 - (c/2)^2}\,t + \phi\right)$

and quasiperiod is $\dfrac{2\pi}{\sqrt{98 - (c/2)^2}}$

As c varies, the quasiperiod can take any value larger than $\dfrac{2\pi}{\sqrt{98}} = \dfrac{2\pi}{7\sqrt{2}} \approx .635$

So $\boxed{\text{quasiperiod} \geq \frac{2\pi}{7\sqrt{2}} \approx .635}$

Name:____

11. Find the *complex* solution to the *complex* problem
$y''-(2+2i)y'+(-1+2i)y=0, \quad y(0)=2, y'(0)=2+2i$.

$$\left[D^2-(2+2i)D+(-1+2i)\right]y=0$$

Quadratic Formula $\dfrac{2+2i \pm \sqrt{(2+2i)^2-4(-1+2i)}}{2}$

$$= \frac{2+2i \pm \sqrt{\cancel{8i}+4-\cancel{8i}}}{2}$$

$$= \frac{2+2i \pm 2}{2} \quad \text{roots } i,\ 2+i$$

$$y = c_1 e^{ix} + c_2 e^{(2+i)x}$$

$$y' = ic_1 e^{ix} + (2+i)c_2 e^{(2+i)x}$$

So $y(0) = c_1 + c_2 \overset{\text{set}}{=} 2$

$y'(0) = ic_1 + (2+i)c_2 \overset{\text{set}}{=} 2+2i$ $(\times i)$

$$ic_1 + (2+i)c_2 = 2+2i$$
$$-(ic_1 + ic_2 = 2i)$$
$$2c_2 = 2$$
$$c_2 = 1$$
$$c_1 = 1$$

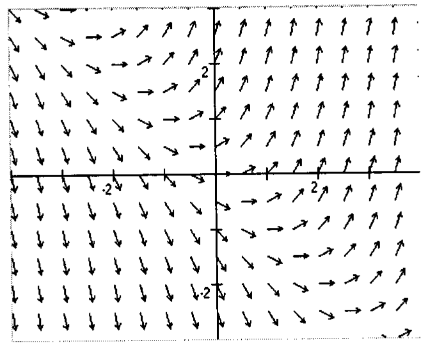
$$\boxed{y = e^{ix} + e^{(2+i)x}}$$

Name:________________________________

12. Match the following equations with their slope fields.

(A) $\frac{dy}{dx} = y - x$

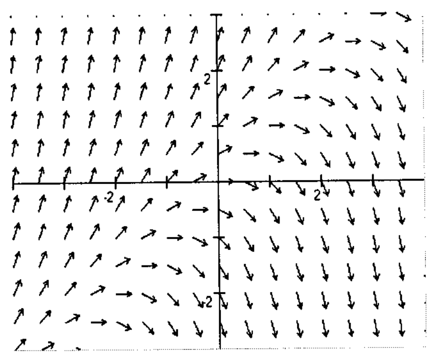
(i)



Slope = 0 when ~~x = y~~ y = −x

(B) $\frac{dy}{dx} = x + y$

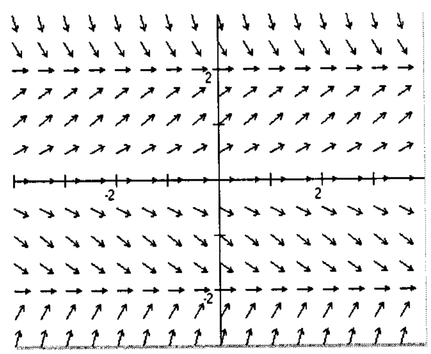
(ii)



Slope = 0 when y = x

(C) $\frac{dy}{dx} = -.5xy$

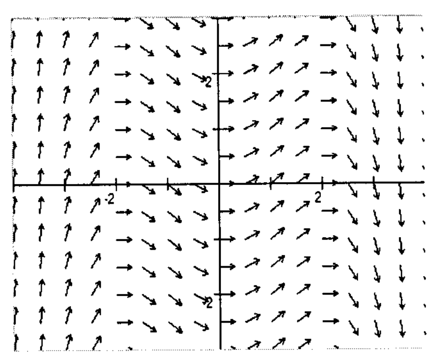
(iii)



Slope depends only on y

(D) $\frac{dy}{dx} = x - .25x^3$

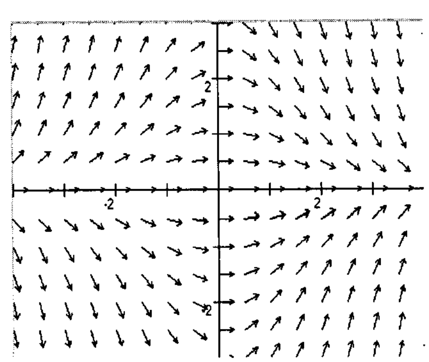
(iv)



Slope depends only on x

(E) $\frac{dy}{dx} = y - .25y^3$

(v)



Slope = 0 when x = 0 or y = 0