The Lie bialgebra of loops on surfaces of Goldman and Turaev

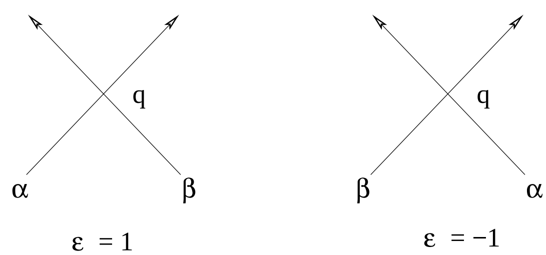
Consider any surface F. Let $\hat{\pi}$ be the set of free homotopy classes of loops on F. Let Z be the vector space with basis $\hat{\pi} \setminus \{\text{null}\}$ where "null" is the homotopy class of the null loop. We consider the null loop to be 0 in Z.

For any two loops α, β in general position on F, let $\alpha \cap \beta$ denote the set of (simple) intersections of α and β.

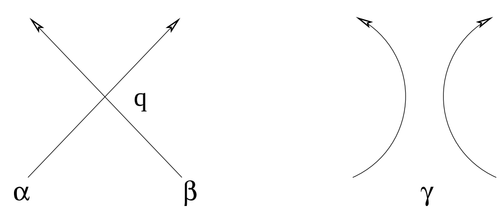
We can give Z a Lie algebra structure by taking two loops, and summing over their intersections of the loop obtained by removing the intersection and smoothing. Precisely, we get the formula

$$[\langle a\rangle, \langle b\rangle] = \sum_{q\in\alpha\cap\beta} \epsilon(q;\alpha,\beta)\langle\alpha_q\beta_q\rangle \tag{0.1}$$

where $\epsilon(q;\alpha,\beta) = \pm 1$ as determined by the figure



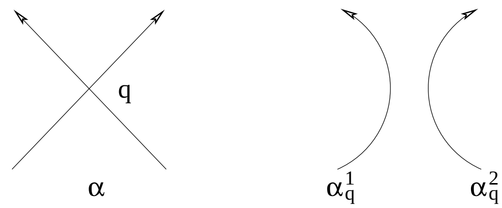
and $\alpha_q\beta_q$ is the product of the loops α, β **based at** q**.** That is, $\langle\alpha_q\beta_q\rangle = \langle\gamma\rangle$ where γ is the "smoothed" version



The coproduct is given by summing over self-intersections and removing the intersection point, and smoothing to obtain two paths. Precisely, let $\cap\alpha$ denote the set of self-intersections of α (assuming α is in general position). The coproduct is then given by summing over self-intersections and removing the intersection point, and smoothing to obtain two paths. Then

$$\delta(\langle\alpha\rangle) = \sum_{q\in\cap\alpha} \langle\alpha_q^1\rangle \wedge \langle\alpha_q^2\rangle \quad (a\wedge b := a\otimes b - b\otimes a) \tag{0.2}$$

where α_q^1, α_q^2 are the two pieces of α determined by "cutting" α at q as follows:


q
α
α_q^1
α_q^2

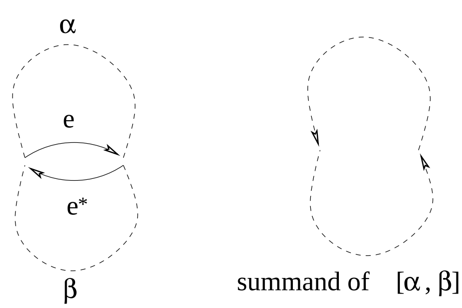
Analogue: the necklace Lie algebra of Ginzburg, Bocklandt, and Le Bruyn

This is essentially replacing the surface with the double of a quiver. Take a quiver Q (a collection of vertices and directed edges between them, where multiple edges with the same endpoints are allowed) and consider, instead of $\hat{\pi}$, the set of loops in the double quiver $\overline{Q} = Q \sqcup Q^*$ (here Q^* replaces every edge $e \in Q$ with the reverse edge e^*, reversing orientation). As a vector space, our Lie algebra L has as a basis this set of loops (without a preferred choice of initial arrow).

More precisely, let Q be a quiver, which we consider the set of arrows, and let I be the set of vertices. Let $\overline{Q}$ be the double quiver (for each $e \in Q$, e^* is the reversed edge). Let $E_{\overline{Q}}$ be the vector space with basis $\overline{Q}$. Let $B = \mathbb{C}^I$ be the semisimple ring with the set I as idempotents. We give $E_{\overline{Q}}$ a B-bimodule structure as follows: for each arrow e let e_s be the starting vertex, and e_t be the terminal vertex. Then we set $e_s e = e = e e_t$ for each $e \in \overline{Q}$, while $ve = 0 = ew$ for any $v \neq e_s, w \neq e_t$. Let P be the path algebra in $\overline{Q}$, i.e. $P = T_B E_{\overline{Q}}$.

Then the Lie algebra $L := P/[P, P]$ where $[P, P]$ is the commutator in the associative algebra P. Note that any path which is not a loop is killed in this quotient, and loops lose their choice of initial arrow.

Now we define the Lie bracket structure as follows. Instead of intersections of two paths, we consider pairs of edges e in one of the loops and e^* in the other, and consider that an "intersection." Then we join the two paths together as before by creating a γ which smooths the two, in this case by removing the e and e^* and joining the edges. If we consider the e and e^* "beads" that we cut the two "necklaces" at, and then join the corresponding ends to form a new necklace, we get the picture which gives this Lie algebra its name:

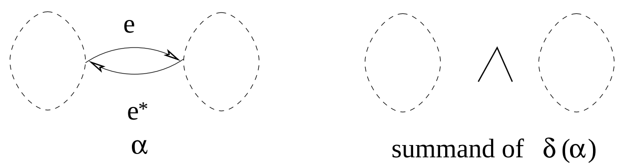


Precisely, we have

$$[a_1 a_2 \cdots a_k, b_1 b_2 \cdots b_l] = \sum_{i,j} [a_i, a_j](a_i)_t a_{i+1} \cdots a_{i-1} b_{j+1} \cdots b_{j-1}, \tag{0.3}$$

$$[a_i, a_j] = \begin{cases} 1 & a_j = (a_i)^*, a_i \in Q \\ -1 & a_i = (a_j)^*, a_j \in Q \\ 0 & \text{otherwise} \end{cases} \tag{0.4}$$

The coproduct is given as follows. Again, instead of self-intersections, we consider pairs of arrows e, e^* in a path, and we then "smooth" by removing these two arrows, creating two paths which are wedged together:

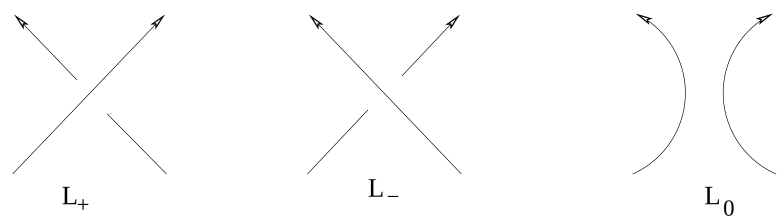


Precisely, we have

$$\delta(a_1 \cdots a_n) = \sum_{i<j} [a_i, a_j](a_j)_t a_{j+1} \cdots a_{i-1} \wedge (a_i)_t a_{i+1} \cdots a_{j-1} \tag{0.5}$$

Turaev's quantization of the Lie bialgebra of loops

The quantization of Z is based on a skein algebra which is closely tied to the Jones polynomial. Take a three-manifold M. Define a *Conway triple* to be a triple of links (L_+, L_-, L_0) in M where all three links are identical outside of a ball, and inside the ball the three look like



Then we let A be the $\mathbb{C}[\mathrm{h}]$-module generated by the set of isotopy classes of links $\mathcal{L}$ modulo isotopy classes of unlinked nullhomotopic loops, and modulo the relations

$$\langle L_+\rangle - \langle L_-\rangle - \mathrm{h}^{\epsilon}\langle L_0\rangle \tag{0.6}$$

for any Conway triple (L_+, L_-, L_0), with ϵ set as follows:

$$\epsilon = \begin{cases} 0 & \#(L_0) = \#(L_+) - 1 = \#(L_-) - 1, \\ 1 & \#(L_0) = \#(L_+) + 1 = \#(L_-) + 1. \end{cases} \tag{0.7}$$

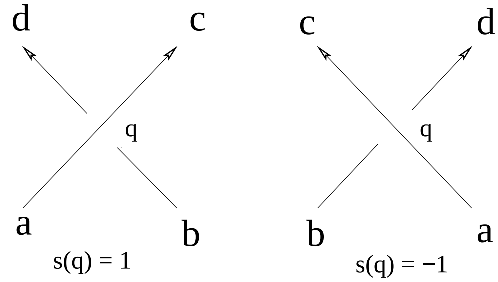
This gives a "skein $\mathbb{C}[\mathrm{h}]$-module."

Now, to define a bialgebra, we need to restrict to the case $M = F \times [0,1]$ where F is a surface. Then we get

$$LL' = \{(a,t) \in F \times [0,1] \mid t \geq \frac{1}{2} \text{ and } (a, 2t-1) \in L, \text{or } t \geq \frac{1}{2} \text{ and } (a, 2t) \in L'\}, \tag{0.8}$$

that is, we stack L "on top of" L'.

To define the coproduct, we need to define "colorings". Given a link L, we make a link diagram of a link isotopic to L with only simple crossings. Then, we color L with two colors, 1 and 2, such that any segment between crossings has a single color, and at each crossing q we have



where either $a = c, b = d$ (each strand has a solid color), or else $a = d > b = c$. In the latter case the intersection is called a *color-cutting intersection*. Let f be the number of color-cutting intersections, and let f_- be the number of color-cutting intersections of negative sign $s(q)$. Let L_i be the subset of L of color i for $i \in \{1, 2\}$; L_i is also a link. Here we "smooth" the color-cutting intersections by changing from an L_+/L_- type crossing to an L_0-type one. Finally, let $N = \#(L) - \#(L_1) - \#(L_2)$.

Now, the coproduct is defined by

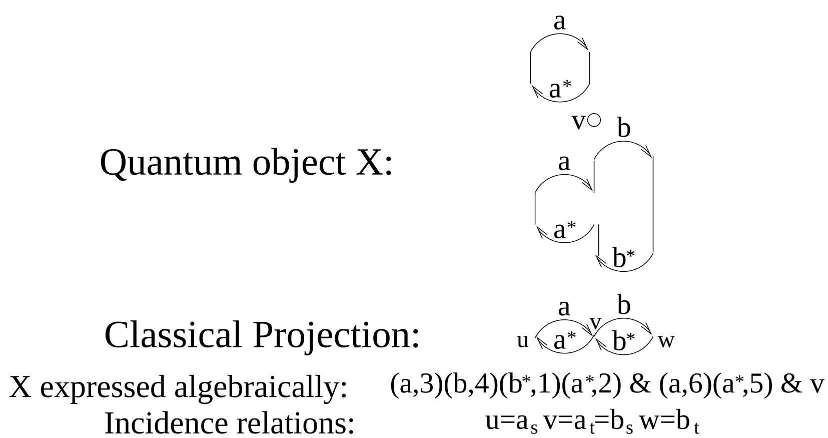
$$\Delta(L) = \sum_{\text{colorings}} (-1)^{f_-} h^{\frac{f+N}{2}} L_1 \otimes L_2. \tag{0.9}$$

The coassociativity property follows readily from showing that $(1 \otimes \Delta)\Delta = (\Delta \otimes 1)\Delta = \Delta^2$, where Δ^2 is defined similarly to Δ but with three colors. Well-definition follows from case-by-case analysis at an intersection. The bialgebra condition follows by seeing that colorings of the product of two links must be the product of colorings on each one (because of the coloring rule). I do not know whether there is an antipode or whether the PBW property is satisfied for this algebra (I have not given it a great deal of thought and I didn't see it in Turaev's paper).

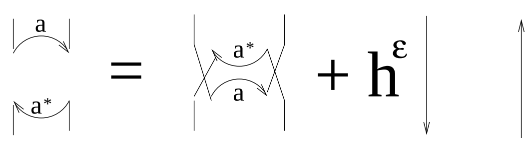
The quantization of the necklace Lie algebra

We define a Hopf algebra emulating Turaev's construction, using our dictionary between loops on surfaces and loops in quivers, between simple crossings and pairs of an arrow in Q and its reverse in Q^*.

Following is a diagram of a quantum element:



We will mod out by relations of the following form, where $\epsilon \in \{0,1\}$:



Precisely, consider the space of "arrows with heights," $\mathrm{AH} := \overline{Q} \times \mathbb{N}$. Let $E_{\overline{Q},\mathrm{h}} = \mathbb{C}\langle \mathrm{AH} \rangle$ be the vector space with basis AH. This can also be viewed as a B-module by $e_s(e,h) = (e,h) = (e,h)e_t$, and $v(e,h) = 0 = (e,h)w$ for any $v \neq e_s, w \neq e_t$. Let $\mathrm{LH} := T_B E_{\overline{Q},\mathrm{h}} / [T E_{\overline{Q},\mathrm{h}}, T E_{\overline{Q},\mathrm{h}}]$ to be the space of cyclic words in AH which form paths once heights are forgotten. There is a canonical projection $\mathrm{LH} \to L$ given by forgetting the heights.

Let $S\mathrm{LH}[h]$ be the symmetric algebra in $\mathrm{LH}[h]$ (this is not over B). A "monomial" here has the form

$$(a_{1,1}, h_{1,1}) \cdots (a_{1,l_1}, h_{1,l_1}) \& (a_{2,1}, h_{2,1}) \cdots (a_{2,l_2}, h_{2,l_2}) \& \cdots \& (a_{k,1}, h_{k,1}) \cdots (a_{k,l_k}, h_{k,l_k}) \& v_1 \& \cdots \& v_m, \tag{0.10}$$

with $a_{i,j} \in \overline{Q}, v_i \in I \subset B$. Let $\tilde{A}$ be the subquotient obtained by only considering monomials where every arrow is at a distinct height (all $h_{i,j}$ are distinct), and so that we don't care what the actual heights are, only the order in which they appear. (So we think of this as creating a link by putting each arrow at a discrete height, connected together by vertical segments).

We mod out by the relations

$$X - X'_{i,j,i',j'} - h^{\delta_{i,i'}} X''_{i,j,i',j'}, \quad \text{where } i \neq i', h_{i,j} < h_{i',j'}, \text{ and } \nexists(i'', j'') \text{ with } h_{i,j} < h_{i'',j''} < h_{i',j'} \tag{0.11}$$

where we think of X' as the opposite crossing, and X'' as the "smoothing" analagous to the smoothing we did in Turaev's bialgebra. Precisely, when $i \neq i'$, $X'_{i,j,i',j'}$ is the same as X but with the heights $h_{i,j}$ and $h_{i',j'}$ interchanged, and $X''_{i,j,i',j'}$ replaces the components $(a_{i,1}, h_{i,1}) \cdots (a_{i,l_i}, h_{i,l_i})$ and $(a_{i',1}, h_{i',1}) \cdots (a_{i',l_{i'}}, h_{i',l_{i'}})$ with the single component

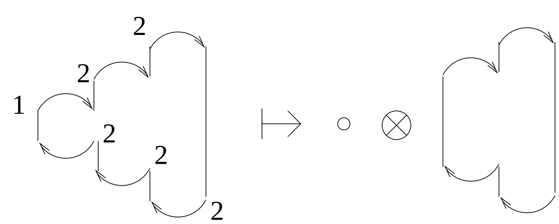
$$[a_{i,j}, a_{i',j'}](a_{i,j})_t (a_{i,j+1}, h_{i,j+1}) \cdots (a_{i,j-1}, h_{i,j-1})(a_{i',j'+1}, h_{i',j'+1}) \cdots (a_{i',j'-1}, h_{i',j'-1}). \tag{0.12}$$

Similarly, $X'_{i,j,i,j'}$ is the same as X but with the heights $h_{i,j}$ and $h_{i,j'}$ interchanged, and $X''_{i,j,i,j'}$ is given by replacing the component $(a_{i,1}, h_{i,1}) \cdots (a_{i,l_i}, h_{i,l_i})$ with the two components

$$[a_{i,j}, a_{i,j'}](a_{i,j'})_t (a_{i,j'+1}, h_{i,j'+1}) \cdots (a_{i,j-1}, h_{i,j-1}) \& (a_{i,j})_t (a_{i,j+1}, h_{i,j+1}) \cdots (a_{i,j'-1}, h_{i,j'-1}). \tag{0.13}$$

Our quantization A of L is given, as a module, by quotienting $\tilde{A}$ by the relations (0.11). The product is again given by putting one loop on top of the other. That is, XX' has the heights of X followed by the heights of X', which are all greater than the heights of X but both are in the same order. The identity is the empty loop, i.e. the one coming from $1 \in S$LH.

The coproduct is given again by colorings: we sum over all ways of picking arrow and reversed-arrow pairs to be "color-cutting intersections", and we "smooth" each color-cutting intersection by removing the reversed arrow pairs, while multiplying in an appropriate sign and powers of h. Then the color-1 part is tensored by the color-2 part with the aforementioned coefficient. Following is an example:



Let $P := \{(i,j) \mid 1 \leq i \leq k, 1 \leq j \leq l_i\}$ be our set of *pairs*, which has addition defined by

$$(i,j) + j' := (i, j+j') := (i, j''), \quad \text{where } 1 \leq j'' \leq l_i \text{ and } j + j' \equiv j'' \pmod{l_i}. \tag{0.14}$$

We also use the notation $a_{(i,j)} := a_{i,j}$ and $h_{(i,j)} := h_{i,j}$. Now, we choose a set of pairs $I \subset P$ together with a self-pairing $\phi : I \to I$ that is involutive and has no fixed points, and which satisfies the condition:

For any $(i,j) \in I$ where $\phi(i,j) = (i',j')$, we have $l_i, l_j > 0$ and $a_{i,j} = a^*_{i',j'}$.

Also, let $V = \{1, \ldots, m\}$ correspond to the vertex idempotents in (0.10). Then, a coloring of X *with* (I,ϕ)*-cutting pairs* is a mapping $c : P \sqcup V \to \{1,2\}$ satisfying the conditions:

(1) For each $(i,j) \in P \setminus I$, we have $c(i,j) = c(i,j+1)$; (2) for each $(i,j) \in I$, we have $c(i,j) = c(\phi(i,j)+1) \neq c(i,j+1) = c(\phi(i,j))$, and we have $c(i,j) > c(\phi(i,j))$ iff $h_{i,j} > h_{\phi(i,j)}$.

Given a coloring (I, ϕ, c), we define a map $f : P \to P$ by $f(i,j) = (i,j)+1$ if $(i,j) \notin I$ and $f(i,j) = \phi(i,j)+1$ otherwise. Note that in the latter case, $a_{i,j} = a_{\phi(i,j)+1}$. Note also that f is invertible by $f^{-1}(i,j) = (i,j) - 1$ if $(i,j) - 1 \notin I$ and $f^{-1}(i,j) = \phi((i,j)-1)$ otherwise. Then we can partition P into orbits under f, $P = P_1 \sqcup \cdots \sqcup P_q$. Also, note that each orbit P_i is monochrome: $c(P_i) = \{t\}$ for some $1 \leq t \leq n$. For each orbit P_i we define a corresponding element Y_i of A as follows: Suppose $P_i = \{x_1, \ldots, x_p\} \subset P$ for $f(x_i) = x_{i+1}$ and $f(x_p) = x_1$. Then, for each x_i we let $y_i = (a_{x_i}, h_{x_i})$ if $x_i \notin I$ and $y_i = (a_{x_i})_s$ otherwise. Then we set $Y_i = y_1 \cdots y_p \in \mathrm{LH} \subset A$. Let us suppose that the Y_i are arranged so that $Y_1, \ldots, Y_r$ have color 1 and $Y_{r+1}, \ldots, Y_q$ have color 2. Also suppose that we order V so that $v_1, \ldots, v_u$ have color 1 and $v_{u+1}, \ldots, v_m$ have color 2.

To define the sign, partition I into $I_Q = \{(i,j) \in I \mid a_{i,j} \in Q\}$ and $I_{Q^*} = \{(i,j) \in I \mid a_{i,j} \in Q^*\}$, so that $\phi(I_Q) = I_{Q^*}$ and vice-versa. For each $(i,j) \in I_Q$ set $s_{i,j} = 1$ if $h_{i,j} < h_{\phi(i,j)}$ and $s_{i,j} = -1$ otherwise. Now, define the sign $s(I, \phi, c)$ as follows:

$$s(I, \phi, c) = \prod_{(i,j) \in I_Q} s_{i,j}. \tag{0.15}$$

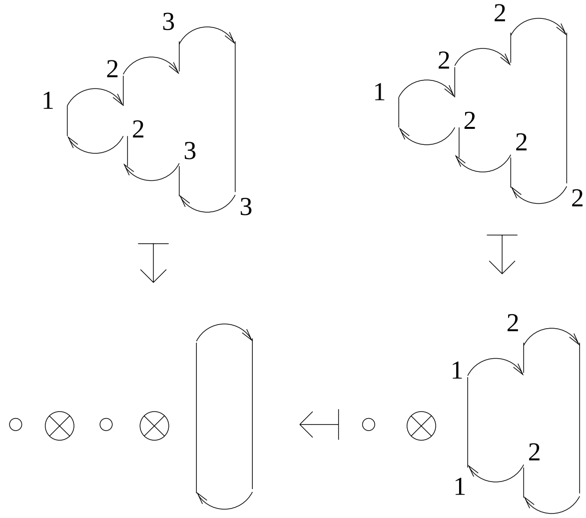
Recalling k from (0.10), we set

$$N' = k - q, \tag{0.16}$$

$$\Delta^{n-1}(X) = \sum_{n-\text{colorings}(I,\phi,c)} s(I, \phi, c) \mathrm{h}^{\#(I)/4 + N'/2} X^1_{I,\phi,c} \otimes \cdots \otimes X^n_{I,\phi,c}, \tag{0.17}$$

extending it $\mathbb{C}[\mathrm{h}]$-linearly to all of A. Here note that $\#(I)/4$ is the same as half the number of color-cutting intersections, and $N' = k - q = k - u - (r - u)$ is the analogue to the previous N'.

As before, coassociativity is proved by defining colorings on three colors:



well-definition is proved by analysis at an intersection, and the bialgebra condition is proved by showing that colorings of XX' come from products of colorings on X with colorings on X'. By discreteness, it is easy to define an antipode (because color-cutting intersections in coproducts decrease the number of arrows). We also have the PBW property:

Theorem 0.1. *A is isomorphic as a $\mathbb{C}[\mathrm{h}]$-module to $UL \cong SL$, by descending the forgetful map $\tilde{A} \to SL$.*

I proved PBW using a detailed combinatorial argument based on Bergman's Diamond Lemma.

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