How to Construct an $r-$matrix on a Lie Superalgebra

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- $\mathfrak{g}$ a finite dimensional (simple) Lie algebra with a non-degenerate $\mathfrak{g}-$invariant bilinear form $(\ ,\ )$
  $r \in \mathfrak{g} \otimes \mathfrak{g}$

- **Classical Yang-Baxter Equation** for r:

$$[r^{12}, r^{13}] + [r^{12}, r^{23}] + [r^{13}, r^{23}] = 0.$$

  A solution r to the classical Yang-Baxter equation is called a **classical** $r-$**matrix** (or simply an $r-$**matrix**).

- r is called **non-degenerate** if it satisfies:

$$r^{12} + r^{21} \neq 0.$$

BELAVIN-DRINFELD CLASSIFICATION:

The Setup

- $\mathfrak{g}$ simple Lie algebra
- $\Omega \in (\mathfrak{g} \otimes \mathfrak{g})^{\mathfrak{g}}$ corresponding to the quadratic Casimir
- $\mathfrak{b}_+$ fixed positive Borel
- $\mathfrak{h} \subset \mathfrak{b}$ Cartan subalgebra
- $\Delta \subset \mathfrak{h}^*$ roots
- $\Gamma = \{\alpha_1, \alpha_2, \cdots, \alpha_n\}$ a basis for Δ
- $\{X_\alpha, Y_\alpha, H_\alpha | \alpha \in \Gamma\}$ Weyl-Chevalley generators

- $(\Gamma_1, \Gamma_2, \tau)$ **admissible triple** (**Belavin - Drinfeld triple**) if:

  1. $\Gamma_i \subset \Gamma$ and $\tau : \Gamma_1 \to \Gamma_2$ is a bijection;
  2. for any $\alpha, \beta \in \Gamma_1$, $(\tau(\alpha), \tau(\beta)) = (\alpha, \beta)$;
  3. for any $\alpha \in \Gamma_1$ there exists a $k \in \mathbb{N}$ such that $\tau^k(\alpha) \notin \Gamma_1$.

  Let $(\Gamma_1, \Gamma_2, \tau)$ be an admissible triple.

- $\mathfrak{g}_i$ the subalgebra of $\mathfrak{g}$ generated by $\{X_\alpha, Y_\alpha, H_\alpha\}$ $\alpha \in \Gamma_i$.

- $\varphi : \mathfrak{g}_1 \to \mathfrak{g}_2 :$ $\begin{cases} \varphi(X_\alpha) & = & X_{\tau(\alpha)} \\ \varphi(Y_\alpha) & = & Y_{\tau(\alpha)} \\ \varphi(H_\alpha) & = & H_{\tau(\alpha)} \end{cases}$ $\alpha \in \Gamma_1$

  extends (uniquely) to an isomorphism $\varphi : \mathfrak{g}_1 \to \mathfrak{g}_2$

- Extend τ to $\overline{\tau} : \overline{\Gamma}_1 \to \overline{\Gamma}_2$, where $\overline{\Gamma}_i =$ nonnegative integral span of Γ_i in Δ

- Choose a basis $\{e_\alpha | \alpha \in \Delta\}$ for non-Cartan $\mathfrak{g}$:

  1. $e_\alpha \in \mathfrak{g}_\alpha$ for each α;
  2. $(e_\alpha, e_{-\alpha}) = 1$ for any α;
  3. $\varphi(e_\alpha) = e_{\overline{\tau}(\alpha)}$ for all $\alpha \in \overline{\Gamma}_1$

- Partial order on Δ^+:

  $\alpha \prec \beta$ if and only if $\exists k \in \mathbb{N}$ such that $\beta = \overline{\tau}^k(\alpha)$

  If $\alpha \prec \beta$, then necessarily $\alpha \in \overline{\Gamma}_1$,$\beta \in \overline{\Gamma}_2$.

BELAVIN-DRINFELD CLASSIFICATION:

The Theorem

1. If $r_0 \in \mathfrak{h} \otimes \mathfrak{h}$ satisfies
$$r_0^{12} + r_0^{21} = \Omega_0 \tag{1}$$
$$(\tau(\alpha) \otimes 1)(r_0) + (1 \otimes \alpha)(r_0) = 0 \ (\alpha \in \Gamma_1) \tag{2}$$
then $r \in \mathfrak{g} \otimes \mathfrak{g}$ defined by:
$$r = r_0 + \sum_{\alpha > 0} e_{-\alpha} \otimes e_\alpha + \sum_{\alpha,\beta>0,\alpha\prec\beta} (e_{-\alpha} \otimes e_\beta - e_\beta \otimes e_{-\alpha})$$
is a solution to the system:
$$r^{12} + r^{21} = \Omega$$
$$[r^{12}, r^{13}] + [r^{12}, r^{23}] + [r^{13}, r^{23}] = 0$$

2. Any solution to this system can be obtained as above from some admissible triple $(\Gamma_1, \Gamma_2, \tau)$ and some $r_0 \in \mathfrak{h} \otimes \mathfrak{h}$ that satisfies Equations 1 and 2, by choosing a suitable triangular decomposition of $\mathfrak{g}$ and a set of Weyl-Chevalley generators.

Objective: To develop a similar theory for super structures.

- A **Lie superalgebra** is a superalgebra $\mathfrak{g} = \mathfrak{g}_{\overline{0}} \oplus \mathfrak{g}_{\overline{1}}$ where the algebra operation is called a **super bracket**, denoted by $[\cdot, \cdot]$, and satisfies the following conditions:

  1. $[g_1, g_2] = -(-1)^{|g_1||g_2|}[g_2, g_1]$;
  2. $(-1)^{|g_1||g_3|}[[g_1, g_2], g_3] + (-1)^{|g_2||g_1|}[[g_2, g_3], g_1] +$ $(-1)^{|g_3||g_2|}[[g_3, g_1], g_2] = 0$

  where each g_i is homogeneous of parity $|g_i|$.

- Main example:

  $$sl(m, n) = \{A \in gl(m, n) | str(A) = 0\}$$

- $\mathfrak{g} = \mathfrak{g}_{\bar{0}} \oplus \mathfrak{g}_{\bar{1}}$ simple Lie superalgebra with nondegenerate Killing form $(\cdot, \cdot)$

- $\Omega \in (\mathfrak{g} \otimes \mathfrak{g})^{\mathfrak{g}}$ corresponding to the quadratic Casimir

- $\mathfrak{h} \subset \mathfrak{g}$ a Cartan subsuperalgebra ($\mathfrak{h} \subset \mathfrak{g}_{\bar{0}}$)

- $\Delta = \Delta_{\bar{0}} + \Delta_{\bar{1}}$ roots of $\mathfrak{g}$ associated with $\mathfrak{h}$

- $\mathfrak{b} \supset \mathfrak{h}$ a Borel subsuperalgebra ($\mathfrak{b} = \mathfrak{h} \oplus \bigoplus_{\alpha \in \Delta^+} \mathfrak{g}_\alpha$)

- $\Gamma = \{\alpha_1, \alpha_2, \cdots, \alpha_n\}$ a basis for Δ

- $\{X_\alpha, Y_\alpha, H_\alpha | \alpha \in \Gamma\}$ Weyl-Chevalley generators

Technical Lemma

Let $\mathfrak{g}$ be a simple Lie superalgebra with non-degenerate Killing form $(\cdot,\cdot)$. Let $\Omega \in (\mathfrak{g} \otimes \mathfrak{g})^{\mathfrak{g}}$ be the element corresponding to $(\cdot,\cdot)$. Let $f : \mathfrak{g} \to \mathfrak{g}$ be an even linear map, and set $r = (f \otimes 1)\Omega$.

Then the system of equations:

$$r^{12} + r^{21} = \Omega$$

$$[r^{12}, r^{13}] + [r^{12}, r^{23}] + [r^{13}, r^{23}] = 0$$

is equivalent to the system:

$$f + f^* = 1$$

$$(f-1)[f(x), f(y)] = f([(f-1)(x), (f-1)(y)])$$

where f^* stands for the adjoint of f with respect to $(\cdot,\cdot)$.

- $(\Gamma_1, \Gamma_2, \tau)$ **admissible triple** (**Belavin - Drinfeld triple**) if:

  1. $\Gamma_i \subset \Gamma$ and $\tau : \Gamma_1 \to \Gamma_2$ is a bijection;
  2. for any $\alpha, \beta \in \Gamma_1$, $(\tau(\alpha), \tau(\beta)) = (\alpha, \beta)$;
  3. for any $\alpha \in \Gamma_1$ there exists a $k \in \mathbb{N}$ such that $\tau^k(\alpha) \notin \Gamma_1$.
  4. τ preserves the grading of the root space.

  Let $(\Gamma_1, \Gamma_2, \tau)$ be an admissible triple.

- $\mathfrak{g}_i$ the subalgebra of $\mathfrak{g}$ generated by $\{X_\alpha, Y_\alpha, H_\alpha\}$ $\alpha \in \Gamma_i$.

- $\varphi : \mathfrak{g}_1 \to \mathfrak{g}_2 :$ $\begin{cases} \varphi(X_\alpha) & = & X_{\tau(\alpha)} \\ \varphi(Y_\alpha) & = & Y_{\tau(\alpha)} \\ \varphi(H_\alpha) & = & H_{\tau(\alpha)} \end{cases}$ $(\alpha \in \Gamma_1)$

  extends (uniquely) to an isomorphism $\varphi : \mathfrak{g}_1 \to \mathfrak{g}_2$

- Extend τ to $\overline{\tau} : \overline{\Gamma}_1 \to \overline{\Gamma}_2$, where $\overline{\Gamma}_i =$ nonnegative integral span of Γ_i in Δ

- Choose a basis $\{e_\alpha | \alpha \in \Delta\}$ for non-Cartan $\mathfrak{g}$:

  1. $e_\alpha \in \mathfrak{g}_\alpha$ for each α;
  2. $(e_\alpha, e_{-\alpha}) = 1$ for any $\alpha \in \Delta^+$;
  3. $\varphi(e_\alpha) = e_{\overline{\tau}(\alpha)}$ for all $\alpha \in \overline{\Gamma}_1$

- Partial order on Δ^+:

  $\alpha \prec \beta$ if and only if $\exists k \in \mathbb{N}$ such that $\beta = \overline{\tau}^k(\alpha)$

  If $\alpha \prec \beta$, then necessarily $\alpha \in \overline{\Gamma}_1$,$\beta \in \overline{\Gamma}_2$.

THEOREM:

Let $r_0 \in \mathfrak{h} \otimes \mathfrak{h}$ satisfy:

$$r_0{}^{12} + r_0{}^{21} = \Omega_0$$

$(\tau(\alpha) \otimes 1)(r_0) + (1 \otimes \alpha)(r_0) = 0$ for all $\alpha \in \Gamma_1$
Then the element r of $\mathfrak{g} \otimes \mathfrak{g}$ defined by:

$$r = r_0 + \sum_{\alpha > 0} e_{-\alpha} \otimes e_\alpha + \sum_{\alpha, \beta > 0, \alpha \prec \beta} (e_{-\alpha} \otimes e_\beta - (-1)^{|\alpha|} e_\beta \otimes e_{-\alpha})$$

is a solution to the system:

$$r + T_s(r) = \Omega$$

$$[r^{12}, r^{13}] + [r^{12}, r^{23}] + [r^{13}, r^{23}] = 0$$

Examples: $\mathfrak{g} = sl(2,1)$

- $\Delta_{\overline{0}} = \{\epsilon_1 - \epsilon_2, \epsilon_2 - \epsilon_1\}$ $\Delta_{\overline{1}} = \{\epsilon_1 - \lambda_1, \epsilon_2 - \lambda_1, \lambda_1 - \epsilon_1, \lambda_1 - \epsilon_2\}$
  where ϵ_i is the (restriction to the Cartan subalgebra of $sl(2,1)$ of the) standard basis: $\epsilon_i(E_{jk}) = \delta_{ij}\delta_{ik}$, and $\lambda_1 = \epsilon_3$. Set $\alpha_1 = \epsilon_1 - \epsilon_2$ and $\alpha_2 = \epsilon_2 - \lambda_1$.

- Six possible Dynkin diagrams D and six possible bases Γ::

  1. $\Gamma(D_1) = \{\alpha_1, \alpha_2\}$.
  2. $\Gamma(D_2) = \{\alpha_1 + \alpha_2, -\alpha_2\}$. (obtained from D_1 via σ_{α_2})
  3. $\Gamma(D_3) = \{-\alpha_1 - \alpha_2, \alpha_1\}$. (obtained from D_2 via $\sigma_{\alpha_1+\alpha_2}$)
  4. $\Gamma(D_4) = -\Gamma(D_1) = \{-\alpha_1, -\alpha_2\}$.
  5. $\Gamma(D_5) = -\Gamma(D_2) = \{-\alpha_1 - \alpha_2, \alpha_2\}$. (obtained from D_4 via $\sigma_{-\alpha_2}$)
  6. $\Gamma(D_6) = -\Gamma(D_3) = \{\alpha_1 + \alpha_2, -\alpha_1\}$. (obtained from D_5 via $\sigma_{-\alpha_1-\alpha_2}$)

The Standard r-matrix

corresponds to $\Gamma_1 = \Gamma_2 = \varnothing, \tau = 0$

1. $r_{st}(D_1) = r_0 + (E_{21} \otimes E_{12}) + (E_{32} \otimes E_{23}) + (E_{31} \otimes E_{13})$
2. $r_{st}(D_2) = r_0 + (E_{21} \otimes E_{12}) - (E_{23} \otimes E_{32}) + (E_{31} \otimes E_{13})$
3. $r_{st}(D_3) = r_0 + (E_{21} \otimes E_{12}) - (E_{23} \otimes E_{32}) - (E_{13} \otimes E_{31})$
4. $r_{st}(D_4) = r_0 + (E_{12} \otimes E_{21}) - (E_{23} \otimes E_{32}) - (E_{13} \otimes E_{31})$
5. $r_{st}(D_5) = r_0 + (E_{12} \otimes E_{21}) + (E_{32} \otimes E_{23}) - (E_{13} \otimes E_{31})$
6. $r_{st}(D_6) = r_0 + (E_{12} \otimes E_{21}) + (E_{32} \otimes E_{23}) + (E_{31} \otimes E_{13})$

All these r-matrices are related to one another via (even or odd) reflections.

Nonstandard r-matrices nontrivial τ: using diagrams D_2 and D_5.

- D_2: $\Gamma_1 = \{\alpha_1 + \alpha_2\}$, $\Gamma_2 = \{-\alpha_2\}$,

  $\tau(\alpha_1 + \alpha_2) = -\alpha_2$:

  $r_{ns_1} = r_{st}(D_2) + ((E_{31} \otimes E_{32}) + (E_{32} \otimes E_{31}))$

- D_5: $\Gamma_1 = \{\alpha_2\}$, $\Gamma_2 = \{-\alpha_1 - \alpha_2\}$,

  $\tau(\alpha_2) = -\alpha_1 - \alpha_2$:

  $r_{ns_2} = r_{st}(D_5) + ((E_{32} \otimes E_{31}) + (E_{31} \otimes E_{32}))$

- D_2: $\tau(-\alpha_2) = \alpha_1 + \alpha_2$:

  $r_{ns_3} = r_{st}(D_2) + (-E_{23} \otimes E_{13}) + (-E_{13} \otimes E_{23})$;

- D_5: $\tau(-\alpha_1 - \alpha_2) = \alpha_2$:

  $r_{ns_4} = r_{st}(D_5) + (-E_{13} \otimes E_{23}) + (-E_{23} \otimes E_{13})$.

Upto isomorphism and super twist $\Rightarrow$ unique nonstandard r.

Question: Can we construct all $r-$matrices this way?

- On $\mathfrak{g} = sl(2,1)$ define a linear map by:

$$\begin{array}{cc} f(E_{11}+E_{33}) = 0 & f(E_{22}+E_{33}) = E_{22}+E_{33} \\ f(E_{21}) = 0 & f(E_{12}) = E_{12} \\ f(E_{23}) = 0 & f(E_{13}) = E_{13} \\ f(E_{31}) = -E_{13} & f(E_{32}) = E_{23}+E_{32} \end{array}$$

- If $r(f) = (f \otimes 1)\Omega$, we get:

$$\begin{aligned} r(f) &= r_0 + E_{12} \otimes E_{21} - E_{13} \otimes E_{31} \\ &+ E_{32} \otimes E_{23} - E_{13} \otimes E_{13} + E_{23} \otimes E_{23} \end{aligned}$$

  where $r_0 = (-E_{22} - E_{33}) \otimes (E_{11} + E_{33})$.

- $Im(f)$ and $Im(f-1)$ will never be simultaneously isomorphic to root subsuperalgebras. The corresponding subsuperalgebras for functions constructed by the theorem will always be root subsuperalgebras.